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Cooling load and COP optimization of an irreversible Carnot refrigerator with spin-1/2 systems

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Abstract
A model of an irreversible quantum refrigerator with working medium consisting of many non-interacting spin-1/2 systems is established in this paper. The quantum refrigeration cycle is composed of two isothermal processes and two irreversible adiabatic processes and is referred to as a spin quantum Carnot refrigeration cycle. Expressions of some important performance parameters, such as cycle period, cooling load and coefficient of performance (COP) for the irreversible spin quantum Carnot refrigerator are derived, and detailed numerical examples are provided. The optimal performance of the quantum refrigerator at high temperature limit is analyzed with numerical examples. Effects of internal irreversibility and heat leakage on the performance are discussed in detail. The endoreversible case, frictionless case and the case without heat leakage are discussed in brief.

Keywords: Finite time thermodynamics; Spin-1/2 systems; Quantum refrigeratoion cycle; Cooling load; COP.

1. Introduction
In recent years, the matrix mechanics developed by Heisenberg, which is an important part of quantum mechanics, has being applied to thermodynamics, and the research object of finite time thermodynamics (FTT) [1-8] has been extended to quantum thermodynamic systems. Considering quantum characteristic of the working medium, many researchers have studied the performance of quantum cycles and obtained many meaningful results. In 1992, Geva and Kosloff [9] first established a quantum heat engine model with working medium consisting of many non-interacting spin-1/2 systems and analyzed the optimal performance of the quantum heat engine using finite time thermodynamic theory. Geva and Kosloff [10] made a comperasion between the spin-1/2 Carnot heat engine and the harmonic Carnot heat engine and indicated that the optimal cycles of spin-1/2 heat engine and harmonic heat engine are not Carnot cycles. Since then, many authors analyzed the performance of endoreversible quantum heat engines using non-interacting harmonic oscillators [11, 12] and spin-1/2 systems [13, 14] as working medium. With rapid development in fields such as aerospace, superconductivity application and infra-red techniques, demands of cryogenic technology are more and more and the investigation relative to quantum refrigerators has attracted a good deal of attention. In 1996, Wu et al [15] first established a quantum Carnot refrigerator model with spin-1/2 systems as working medium and analyzed the optimal performance of the refrigerator. Wu et al [16] analyzed the optimal performance of an endoreversible

quantum Stirling refrigerator with harmonic oscillators as working medium. Several authors analyzed the optimal perfromance of endoreversible quantum Brayton refrigerators [17, 18] with harmonic oscillators [17] and spin-1/2 systems [18] as working medium.

Besides the irreversibility of finite rate heat transfer, other sources of irreversibility, such as the bypass heat leakage, dissipation processes inside the working medium, etc, are considered in performance investigation and optimization on the quantum thermodynamic cycles. In 1996, Jin et al [19] introduced heat leakage between hot reservoir and cold reservoir into exergoeconomic performance optimization of a Carnot quantum engine. In 2000, Feldmann and Kosloff [20] introduced internal friction in the performance investigation for a quantum Brayton heat engine and heat pump with spin-1/2 systems, and the internal friction arose from by non-adiabatic phenomenon on adiabatic branches. Since then, effects of quantum friction on performance of quantum thermodynamic cycles have attracted much more attention [21-27]. Wang et al [25, 26] analyzed the performance of harmonic Brayton [25] and spin-1/2 Brayton [26] heat engines with internal friction and the optimization was performed with respect to the temperatures of the working medium. Considering the inherent regenerative loss, some other authors analyzed effects of non-perfect regeneration on the performances of irreversible spin-1/2 Ericsson refrigerator [28] and irreversible harmonic Stirling refrigerator [29]. Considering heat resistance, non-perfect regeneration, heat leakage and internal irreversibility, Wu et al [30-33] established general irreversible models of quantum Brayton harmonic heat engine [30] and refrigerator [31] as well as quantum spin Carnot heat engine [32] and Ericsson refrigerator [33], and analyzed the effects of the irreversibilities on the performance of the quantum engines and refrigerators. Liu et al [34, 35] established models of general irreversible quantum Carnot heat engines with harmonic oscillators [34] and spin-1/2 systems [35], by taking accounting irreversibilities of heat resistance, internal friction and bypass heat leakage, and studied the optimal ecological performances of the quantum heat engines.

Besides performance of harmonic and spin-1/2 quantum refrigeration cycles, many authors studied the performance of quantum refrigerator using ideal quantum Bose and Fermi gases [36-38]. Bartana and Kosloff [39] and Wu et al [40] studied the thermodynamic performance of laser cryocoolers. Palao and Kosloff [41] established a there-level molecular cooling cycle model and obtained the dependence of the maximum attainable cooling load on temperature at ultra-low temperatures. Some authors studied the performance of irreversible quantum magnetic refrigerators [42-44]. Kosloff and Geva [45] analyzed a three-level quantum refrigerator and its irreversible thermodynamic performance as absolute zero is approached. Rezek et al [46] found that a limiting scaling law between the optimal cooling load and temperature $\dot{Q}_c \propto T_c^{\delta}$ quantifies the principle of unattainability of absolute zero.

Based on Refs. [19, 20, 31, 33], this paper will establish a model of an irreversible quantum Carnot refrigerator with working medium consisting of non-interacting spin-1/2 systems. The refrigeration cycle is composed of two isothermal branches and two irreversible adiabatic branches. The irreversibilities of heat resistance between heat reservoirs and working medium, internal friction caused by non-adiabatic phenomenon on adiabatic branches and bypass heat leakage between hot and cold reservoirs are considered. This paper will derive expressions of cycle period, cooling load and COP of the irreversible quantum Carnot refrigerator by using quantum master equation, semi-group approach and finite time thermodynamics. Especially, optimal performance of the refrigerator at high temperature limit will be analyzed. Effects of internal irreversibility and heat leakage on the optimal performance of the quantum refrigerator will be discussed in detail. The results obtained are more general and can provide some guidelines for optimum design of real quantum refrigerators.

2. Dynamic law of a spin-1/2 system

The Hamiltonian of the interaction between a magnetic field $\vec{B}$ and a magnetic moment $\hat{M}$ is given by $\hat{H}(t) = -\hat{M} \cdot \vec{B}$. For a single spin-1/2 system, the Hamiltonian is given by [47, 48]

$$\hat{H}_{\mathrm{S}} = -\hat{M} \cdot \vec{B} = \mu_{\mathrm{B}} \hat{\sigma} \cdot \vec{B} = 2\mu_{\mathrm{B}} \hat{S} \cdot \vec{B} / \hbar = 2\mu_{\mathrm{B}} \hat{S}_z B_z / \hbar \tag{1}$$

where $\hat{\sigma}(\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$ is the Pauli operator, $\hat{S}(\hat{S}_x, \hat{S}_y, \hat{S}_z)$ is the spin operator of the particle, μ_{B} is the Bohr magneton, $\hbar$ is the reduced Planck's constant and $\vec{B} = \vec{B}(t)$ is the magnetic induction (an external magnetic field) along the positive z axis. The directions of $\hat{S}$ and $\hat{M}$ are opposite. As described in Ref.

[9], one can define $\omega(t) = 2\mu_B B(t)_z$ and refer to ω rather than $B(t)_z$ as “the magnetic field” throughout this paper. Thus, the Hamiltonian of an isolated single spin-1/2 system in the presence of the field $\omega(t)$ may be expressed as

$$\hat{H}_S(t) = \omega(t)\hat{S}_z/\hbar \tag{2}$$

The internal energy of the spin-1/2 system is simply the expectation value of the Hamiltonian

$$E_S = \langle \hat{H}_S \rangle = \omega \langle \hat{S}_z \rangle / \hbar = \omega S/\hbar \tag{3}$$

According to statistical mechanics, the expectation value of a spin angular momentum $\hat{S}_z$ is

$$S = \langle \hat{S}_z \rangle = -\frac{\hbar}{2}\tanh(\frac{\beta\omega}{2}) \tag{4}$$

where $-\hbar/2 < S < 0$, $\beta = 1/(k_B T)$, k_B is the Boltzmann constant and T is the absolute temperature of the spin-1/2 system. For simplicity, the “temperature” will refer to β rather than T throughout this paper.
While the spin-1/2 system is thermally coupled to a heat reservoir (bath), it becomes an open system. The total Hamiltonian of the system-bath is given by

$$\hat{H} = \hat{H}_S + \hat{H}_{SB} + \hat{H}_B \tag{5}$$

where $\hat{H}_S$, $\hat{H}_{SB}$ and $\hat{H}_B$ stand for the spin-1/2 system, system-bath and bath Hamiltonians, respectively. Effects of $\hat{H}_{SB}$ and $\hat{H}_B$ on the spin-1/2 system are included in the Heisenberg equation as additional relaxation-type terms for the system operators. Using the master equation and in the Heisenberg picture, one can obtain the motion of an operator

$$\frac{d\hat{X}}{dt} = \frac{i}{\hbar}\left[\hat{H}_S, \hat{X}\right] + \frac{\partial \hat{X}}{\partial t} + L_D(\hat{X}) \tag{6}$$

where $L_D(\hat{X})$ is a dissipation term (the relaxation term) which originates from a thermal coupling of the spin-1/2 system to a heat reservoir. The system-bath coupling is further assumed to be represented in the form

$$\hat{H}_{SB} = \sum_\alpha \Gamma_\alpha \hat{Q}_\alpha \hat{B}_\alpha \tag{7}$$

where $\hat{Q}_\alpha$ is an operator of the spin-1/2 system, $\hat{B}_\alpha$ is an operator of the bath, and Γ_α is an interaction strength operator. Using semi-group approach, one can obtain [49, 50]

$$L_D(\hat{X}) = \sum_\alpha \gamma_\alpha (\hat{Q}_\alpha^+ \left[\hat{X}, \hat{Q}_\alpha\right] + \left[\hat{Q}_\alpha^+, \hat{X}\right]\hat{Q}_\alpha) \tag{8}$$

where $\hat{Q}_\alpha$ and $\hat{Q}_\alpha^+$ are operators in the Hilbert space of the system and Hermitian conjugates, and γ_α are phenomenological positive coefficients.
Substituting $\hat{X} = \hat{H}_S = \omega\hat{S}/\hbar$ into equation (6) yields

$$\frac{dE_S}{dt} = \frac{d}{dt}\langle \hat{H}_S \rangle = \left\langle \frac{\partial \hat{H}_S}{\partial t} \right\rangle + \langle L_D(\hat{H}_S) \rangle = S\frac{d\omega}{dt}\Big/\hbar + \omega\frac{dS}{dt}\Big/\hbar \tag{9}$$

Comparing with the differential form of the first law of thermodynamics

$$\frac{dE_S}{dt} = \frac{dW}{dt} + \frac{dQ}{dt} \tag{10}$$

One can easily find that the instantaneous power and inexact differential of work may be identified by

$$P = \left\langle \partial \hat{\mathrm{H}}_{\mathrm{S}} / \partial t \right\rangle = \dot{\omega} S / \hbar = \mathrm{d}W/\mathrm{d}t \tag{11}$$

$$\mathrm{d}W = S\mathrm{d}\omega/\hbar \tag{12}$$

The instantaneous heat flow and inexact differential of heat may be identified by

$$\dot{Q} = \left\langle L_{\mathrm{D}}(\hat{\mathrm{H}}_{\mathrm{S}}) \right\rangle = \omega \dot{S}/\hbar = \mathrm{d}Q/\mathrm{d}t \tag{13}$$

$$\mathrm{d}Q = \omega \mathrm{d}S/\hbar \tag{14}$$

It is thus clear that, for a spin-1/2 system, equation (9) gives the time derivative of the first law of thermodynamics.

For a spin-1/2 system, $\hat{Q}_\alpha^+$ and $\hat{Q}_\alpha$ are chosen to be the spin creation and annihilation operators: $\hat{S}_+ = \hat{S}_x + i\hat{S}_y$ and $\hat{S}_- = \hat{S}_x - i\hat{S}_y$. Substituting $\hat{S}_+$ and $\hat{S}_-$ into equation (6) and using $\left[\hat{S}_x, \hat{S}_y\right] = i\hbar\hat{S}_z$, $\left[\hat{S}_y, \hat{S}_z\right] = i\hbar\hat{S}_x$, $\left[\hat{S}_z, \hat{S}_x\right] = i\hbar\hat{S}_y$ and $\hat{S}_x^2 = \hat{S}_y^2 = \hat{S}_z^2 = \frac{\hbar^2}{4}$ yields

$$\dot{S} = -2\hbar^2(\gamma_+ + \gamma_-)S - \hbar^3(\gamma_- - \gamma_+) \tag{15}$$

If ω is a constant, γ_+ and γ_- are also constants and the solution of equation (15) is given by

$$S(t) = S_{\mathrm{eq}} + [S(0) - S_{\mathrm{eq}}]e^{-2(\gamma_+ + \gamma_-)t} \tag{16}$$

where $S(0)$ is the initial value of S and $S_{\mathrm{eq}} = -\frac{\hbar}{2}\frac{\gamma_- - \gamma_+}{\gamma_- + \gamma_+}$ is the asymptotic value of S. This asymptotic spin angular momentum must correspond to that at thermal equilibrium $S_{\mathrm{eq}} = -\frac{\hbar}{2}\tanh(\frac{\beta\omega}{2})$. Comparison of these two expressions for S_{eq} yields $\gamma_-/\gamma_+ = e^{\beta\omega}$. It is assumed that

$$\gamma_+ = \mathrm{a}e^{\mathrm{q}\beta\omega} \tag{17}$$

$$\gamma_- = \mathrm{a}e^{(1+\mathrm{q})\beta\omega} \tag{18}$$

where a and q are constants, and explicit expressions for γ_+ and γ_- can be obtained in weak-coupling limit in terms of correlation functions of the bath [47]. $\gamma_+, \gamma_- > 0$ requires $\mathrm{a} > 0$. If $\beta\omega \to \infty$, $\gamma_+ \to 0$ and $\gamma_- \to \infty$ hold, it requires $0 > \mathrm{q} > -1$. Substituting equations (17) and (18) into equation (15) yields

$$\dot{S} = -\mathrm{a}\hbar^2 e^{\mathrm{q}\beta\omega}[2(1+e^{\beta\omega})S + \hbar(e^{\beta\omega} - 1)] \tag{19}$$

3. Model of an irreversible spin-1/2 Carnot refrigerator

The working medium of the refrigerator consists of many non-interacting spin-1/2 systems, and it is a two energy level system. The $S-\omega$ diagram of a Carnot cycle, i.e. two isothermal branches connected by two irreversible adiabatic branches, is shown in Figure 1. The refrigerator operates between a hot reservoir B_{h} at constant temperature T_{h} and a cold reservoir B_c at constant temperature T_{c}. Both the hot and cold reservoirs are thermal phonon systems. The reservoirs are infinitely large and their internal relaxations are very strong, therefore, the reservoirs are assumed to be in thermal equilibrium. In the refrigerator, the spin-1/2 systems are not only coupled thermally to the heat reservoirs but also coupled mechanically to an external "magnetic field". The direction of the external magnetic is fixed and along the positive z axis. The field's magnitude can change over time but is not allowed to reach zero where the two energy levels of the spin-1/2 systems are degenerate.

The spin-1/2 systems are coupled thermally to the heat reservoirs in the two isothermal processes. The "temperature" of the warm working medium in the heat rejection process and cold working medium in the heat addition process are designated as β'_{h} and β'_{c}, respectively. For a refrigerator, the second law of thermodynamics requires $\beta'_{\mathrm{c}} > \beta_{\mathrm{c}} > \beta_{\mathrm{h}} > \beta'_{\mathrm{h}}$. The amounts of heat exchange between the heat reservoirs and the working medium are represented by Q'_{h} and Q'_{c} for processes $4 \to 1$ and $2 \to 3$, respectively. Using equation (14), one can obtain

$$Q'_{\mathrm{h}} = -\frac{1}{\hbar}\int_4^1 \omega \mathrm{d}S = \frac{1}{2}\omega_1 \tanh(\beta'_{\mathrm{h}}\omega_1/2) - \frac{1}{2}\omega_4 \tanh(\beta'_{\mathrm{h}}\omega_4/2) - \frac{1}{\beta'_{\mathrm{h}}}\ln\frac{\cosh(\beta'_{\mathrm{h}}\omega_1/2)}{\cosh(\beta'_{\mathrm{h}}\omega_4/2)} \tag{20}$$

$$Q'_{\mathrm{c}} = \frac{1}{\hbar}\int_2^3 \omega \mathrm{d}S = \frac{1}{2}\omega_2 \tanh(\beta'_{\mathrm{c}}\omega_2/2) - \frac{1}{2}\omega_3 \tanh(\beta'_{\mathrm{c}}\omega_3/2) + \frac{1}{\beta'_{\mathrm{c}}}\ln\frac{\cosh(\beta'_{\mathrm{c}}\omega_3/2)}{\cosh(\beta'_{\mathrm{c}}\omega_2/2)} \tag{21}$$

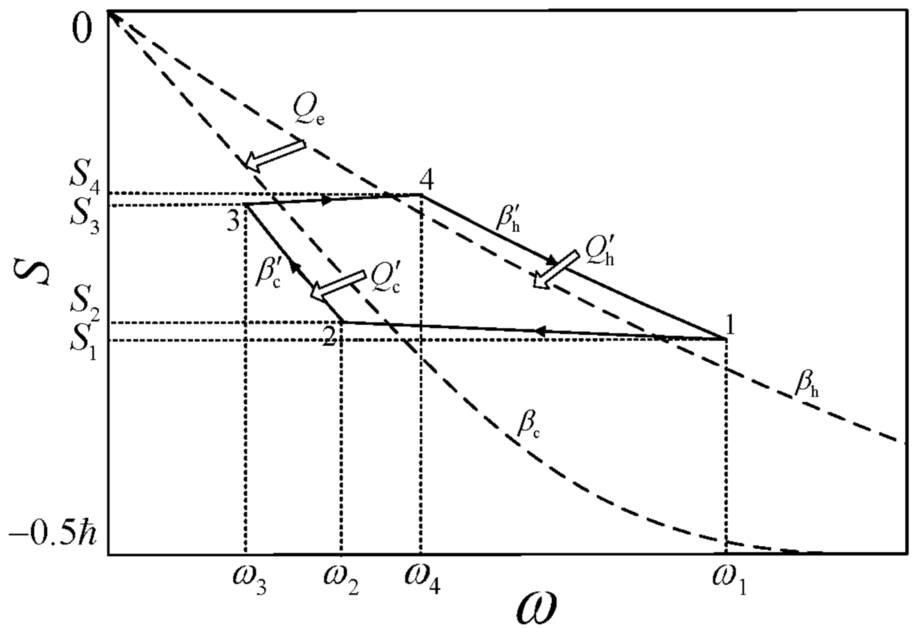

Figure 1. $S-\omega$ diagram of an irreversible quantum Carnot refrigerator cycle with spin-1/2 systems

The working medium system releases heat in the process $4 \to 1$ so that there is a minus before the integral in equation (20). The work done on the system along these processes can be calculated from equation (12)

$$W_{41} = \frac{1}{\hbar}\int_4^1 S\mathrm{d}\omega = \frac{1}{\beta'_{\mathrm{h}}}\ln\frac{\cosh(\beta'_{\mathrm{h}}\omega_4/2)}{\cosh(\beta'_{\mathrm{h}}\omega_1/2)} \tag{22}$$

$$W_{41} = \frac{1}{\hbar}\int_4^1 S\mathrm{d}\omega = \frac{1}{\beta'_{\mathrm{h}}}\ln\frac{\cosh(\beta'_{\mathrm{h}}\omega_4/2)}{\cosh(\beta'_{\mathrm{h}}\omega_1/2)} \tag{23}$$

In adiabatic processes $1 \to 2$ and $3 \to 4$, there are no thermal coupling between working medium and heat reservoirs. It is assumed that the required times of the processes $3 \to 4$ and $1 \to 2$ are τ_{a} and τ_{b}, respectively, and the external magnetic field changes linearly with time, viz.

$$\omega(t) = \omega(0) + \dot{\omega}t \tag{24}$$

According to quantum adiabatic theorem [51], rapid change in the external magnetic field causes quantum non-adiabatic phenomenon. The effect of quantum non-adiabatic phenomenon on the performance characteristics of the refrigerator is similar to effect of internally dissipative friction in the classical analysis. Therefore, one can introduce a friction coefficient μ, which forces a constant speed polarization change, to described non-adiabatic phenomenon, viz.

$$\dot{S} = \hbar(\frac{\mu}{t'})^2 \tag{25}$$

where t' is the time spent on the adiabatic process. Therefore, the spin angular momentum as a function of time is given by [20]

$$S(t) = S(0) + \hbar(\frac{\mu}{t'})^2 t \tag{26}$$

where $0 \le t \le t'$. Substituting $t = \tau_a$ and $t = \tau_b$ into equation (26) yields

$$S_4 = S_3 + \hbar\mu^2/\tau_a \tag{27}$$

$$S_2 = S_1 + \hbar\mu^2/\tau_b \tag{28}$$

where $S_1 = -\frac{\hbar}{2}\tanh\frac{\beta'_h\omega_1}{2}$, $S_2 = -\frac{\hbar}{2}\tanh\frac{\beta'_c\omega_2}{2}$, $S_3 = -\frac{\hbar}{2}\tanh\frac{\beta'_c\omega_3}{2}$ and $S_4 = -\frac{\hbar}{2}\tanh\frac{\beta'_h\omega_4}{2}$ are the spin angular momentums at states 1, 2, 3 and 4, respectively. Combining equations (27) and (28) with equation (4) gives

$$\omega_2 = \frac{2}{\beta'_c}\tanh^{-1}(\tanh\frac{\beta'_h\omega_1}{2} - \frac{2\mu^2}{\tau_b}) \tag{29}$$

$$\omega_4 = \frac{2}{\beta'_h}\tanh^{-1}(\tanh\frac{\beta'_c\omega_3}{2} - \frac{2\mu^2}{\tau_a}) \tag{30}$$

There is no heat exchange between the working medium and heat reservoirs along the adiabatic process, therefore, the work done on the system along processes $3 \to 4$ and $1 \to 2$ can be calculated from equations (3), (24) and (26), respectively

$$W_{34} = \int_0^{\tau_b} \mathrm{d}E_S = \frac{1}{\hbar}\int_0^{\tau_a} S\mathrm{d}\omega + \frac{1}{\hbar}\int_0^{\tau_a} \omega\mathrm{d}S = (\omega_4 - \omega_3)(\frac{S_3}{\hbar} + \frac{\mu^2}{2\tau_a}) + \frac{\mu^2(\omega_3 + \omega_4)}{2\tau_a} \tag{31}$$

$$W_{12} = \int_0^{\tau_b} \mathrm{d}E_S = \frac{1}{\hbar}\int_0^{\tau_b} S\mathrm{d}\omega + \frac{1}{\hbar}\int_0^{\tau_b} \omega\mathrm{d}S = (\omega_2 - \omega_1)(\frac{S_1}{\hbar} + \frac{\mu^2}{2\tau_b}) + \frac{\mu^2(\omega_1 + \omega_2)}{2\tau_b} \tag{32}$$

Besides heat resistance and internal friction, there is heat leakage between hot and cold reservoirs. The heat leakage arises from the coupling action between the hot and cold reservoirs by the working medium of the refrigerator.

The irreversible quantum refrigerator model established in this paper is similar to models of generalized irreversible Carnot refrigerator with classical working medium by taking into account irreversibilities of heat resistance, heat leakage and internal irreversibility [52-56].

4. Cycle period

From (19), one can obtain the expression of time evolution as

$$\tau' = \int_{S_i}^{S_f} \frac{\mathrm{d}S}{\dot{S}} = \int_{\omega_i}^{\omega_f} \frac{\mathrm{d}S/\mathrm{d}\omega}{\dot{S}}\mathrm{d}\omega = -\frac{1}{\mathrm{a}}\int_{\omega_i}^{\omega_f} \frac{(\mathrm{d}S/\mathrm{d}\omega)\mathrm{d}\omega}{\hbar^2 e^{q\beta\omega}[2(e^{\beta\omega}+1)S + \hbar(e^{\beta\omega}-1)]} \tag{33}$$

Equation (33) is a general expression of time evolution for a spin-1/2 system coupling with the heat reservoir and the external magnetic field. So, one can obtain the times of isothermal processes $4 \to 1$ and $2 \to 3$

$$\tau_h = \int_{\omega_4}^{\omega_1} \frac{\mathrm{d}S/\mathrm{d}\omega}{\dot{S}}\mathrm{d}\omega = \frac{1}{2\mathrm{a}\hbar^2}\int_{\beta'_h\omega_4}^{\beta'_h\omega_1} \frac{\mathrm{d}m_h}{e^{q\alpha_h m_h}(e^{\alpha_h m_h} - e^{m_h})(1 + e^{-m_h})} \tag{34}$$

$$\tau_{\rm c}=\int_{\omega_2}^{\omega_3}\frac{{\rm d}S/{\rm d}\omega}{\dot{S}}{\rm d}\omega=\frac{1}{2{\rm a}\hbar^2}\int_{\beta_{\rm c}'\omega_2}^{\beta_{\rm c}'\omega_3}\frac{{\rm d}m_{\rm c}}{e^{{\rm q}\alpha_{\rm c}m_{\rm c}}(e^{\alpha_{\rm c}m_{\rm c}}-e^{m_{\rm c}})(1+e^{-m_{\rm c}})} \tag{35}$$

where $m_{\rm h}=\beta_{\rm h}'\omega$, $m_{\rm c}=\beta_{\rm c}'\omega$, $\alpha_{\rm h}=\beta_{\rm h}/\beta_{\rm h}'$ and $\alpha_{\rm c}=\beta_{\rm c}/\beta_{\rm c}'$.
Consequently, the cycle period is given by

$$\begin{aligned}\tau&=\tau_{\rm h}+\tau_{\rm c}+\tau_{\rm a}+\tau_{\rm b}\\&=\frac{1}{2{\rm a}\hbar^2}\int_{\beta_{\rm h}'\omega_4}^{\beta_{\rm h}'\omega_1}\frac{{\rm d}m_{\rm h}}{e^{{\rm q}\alpha_{\rm h}m_{\rm h}}(e^{\alpha_{\rm h}m_{\rm h}}-e^{m_{\rm h}})(1+e^{-m_{\rm h}})}+\frac{1}{2{\rm a}\hbar^2}\int_{\beta_{\rm c}'\omega_2}^{\beta_{\rm c}'\omega_3}\frac{{\rm d}m_{\rm c}}{e^{{\rm q}\alpha_{\rm c}m_{\rm c}}(e^{\alpha_{\rm c}m_{\rm c}}-e^{m_{\rm c}})(1+e^{-m_{\rm c}})}+\tau_{\rm a}+\tau_{\rm b}\end{aligned} \tag{36}$$

There is heat leakage between hot and cold reservoirs. The hot and cold reservoirs are thermal phonon systems $\rm B_h$ and $\rm B_c$ respectively, and the heat leakage arises from the coupling action between hot and cold reservoirs by the working medium of the refrigerator. The frequency of the thermal phonons of the hot and cold reservoirs are $\omega_{\rm h}$ and $\omega_{\rm c}$, respectively, and the creation and annihilation operators of thermal phonons for hot and cold reservoirs are $\hat{b}_{\rm h}^{+}$, $\hat{b}_{\rm h}^{-}$, $\hat{b}_{\rm c}^{+}$ and $\hat{b}_{\rm c}^{-}$, respectively. The population of the thermal phonons of the cold reservoir is $n_{\rm c}=1/(e^{\hbar\omega_{\rm c}\beta_{\rm c}}-1)$. Similar to $\dot{S}$, one can get derivative of $n_{\rm c}$ as follows at the condition of small thermal disturbance

$$\dot{n}_{\rm c}=-2{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}-1] \tag{37}$$

where $\rm c$ and λ are two constants. From equations (13) and (37), one can get the rate of heat flow from hot reservoir to cold reservoir (i.e. rate of heat leakage) [19]

$$\dot{Q}_{\rm e}=C_{\rm e}\hbar\omega_{\rm c}\dot{n}_{\rm c}=2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1-({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}] \tag{38}$$

where $C_{\rm e}$ is a dimensionless factor connected with the heat leakage. According to the refrigerator model, the hot and cold reservoirs can be assumed to be in thermal equilibrium and $\omega_{\rm c}$, $\beta_{\rm h}$ and $\beta_{\rm c}$ may be assumed to be constants. Therefore, the rate of heat leakage $\dot{Q}_{\rm e}$ is a constants and the heat leakage quantity per cycle is given by

$$Q_{\rm e}=\dot{Q}_{\rm e}\tau=2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1-({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}]\tau \tag{39}$$

5. Cooling load and COP

Combining equations (22), (23), (31) and (32) yields the total work done on the system per cycle

$$\begin{aligned}W_{\rm in}&=\oint{\rm d}W=W_{12}+W_{23}+W_{34}+W_{41}\\&=\frac{1}{\beta_{\rm h}'}\ln\frac{\cosh(\beta_{\rm h}'\omega_4/2)}{\cosh(\beta_{\rm h}'\omega_1/2)}+\frac{1}{\beta_{\rm c}'}\ln\frac{\cosh(\beta_{\rm c}'\omega_2/2)}{\cosh(\beta_{\rm c}'\omega_3/2)}+\frac{(\omega_2-\omega_1)S_1}{\hbar}+\frac{(\omega_4-\omega_3)S_3}{\hbar}+\mu^2(\frac{\omega_2}{\tau_{\rm b}}+\frac{\omega_4}{\tau_{\rm a}})\end{aligned} \tag{40}$$

Combining equations (36) with (40) yields the power input of the refrigerator

$$\begin{aligned}P_{\rm in}&=W_{\rm in}\tau^{-1}\\&=[\frac{1}{\beta_{\rm h}'}\ln\frac{\cosh(\beta_{\rm h}'\omega_4/2)}{\cosh(\beta_{\rm h}'\omega_1/2)}+\frac{1}{\beta_{\rm c}'}\ln\frac{\cosh(\beta_{\rm c}'\omega_2/2)}{\cosh(\beta_{\rm c}'\omega_3/2)}+\frac{(\omega_2-\omega_1)S_1}{\hbar}+\frac{(\omega_4-\omega_3)S_3}{\hbar}+\mu^2(\frac{\omega_2}{\tau_{\rm b}}+\frac{\omega_4}{\tau_{\rm a}})]\tau^{-1}\end{aligned} \tag{41}$$

Combining equations (21), (39) with (36) yields the cooling load of the refrigerator

$$\begin{aligned}R=Q_{\rm c}/\tau&=[\frac{1}{2}\omega_2\tanh(\beta_{\rm c}'\omega_2/2)-\frac{1}{2}\omega_3\tanh(\beta_{\rm c}'\omega_3/2)+\frac{1}{\beta_{\rm c}'}\ln\frac{\cosh(\beta_{\rm c}'\omega_3/2)}{\cosh(\beta_{\rm c}'\omega_2/2)}]\tau^{-1}\\&\quad-2C_{\rm e}{\rm c}\hbar\omega_{\rm c}{\rm e}^{\lambda\hbar\beta_{\rm h}\omega_{\rm c}}[1-({\rm e}^{\hbar\beta_{\rm h}\omega_{\rm c}}-1)n_{\rm c}]\end{aligned} \tag{42}$$

where $Q_{\rm c}=Q_{\rm c}'-Q_{\rm e}$ is the heat released by the cold reservoir. Combining equations (21), (39) with (40) gives the COP of the refrigerator

$$\varepsilon = Q_{\mathrm{c}}/W$$

$$= \frac{\dfrac{1}{2}\omega_2 \tanh(\beta_{\mathrm{c}}'\omega_2/2) - \dfrac{1}{2}\omega_3 \tanh(\beta_{\mathrm{c}}'\omega_3/2) + \dfrac{1}{\beta_{\mathrm{c}}'}\ln\dfrac{\cosh(\beta_{\mathrm{c}}'\omega_3/2)}{\cosh(\beta_{\mathrm{c}}'\omega_2/2)} - 2C_{\mathrm{e}}\mathrm{c}\hbar\omega_{\mathrm{c}}\mathrm{e}^{\lambda\hbar\beta_{\mathrm{h}}\omega_{\mathrm{c}}}[1-(\mathrm{e}^{\hbar\beta_{\mathrm{h}}\omega_{\mathrm{c}}}-1)n_{\mathrm{c}}]\tau}{\dfrac{1}{\beta_{\mathrm{h}}'}\ln\dfrac{\cosh(\beta_{\mathrm{h}}'\omega_4/2)}{\cosh(\beta_{\mathrm{h}}'\omega_1/2)} + \dfrac{1}{\beta_{\mathrm{c}}'}\ln\dfrac{\cosh(\beta_{\mathrm{c}}'\omega_2/2)}{\cosh(\beta_{\mathrm{c}}'\omega_3/2)} + \dfrac{(\omega_2-\omega_1)S_1}{\hbar} + \dfrac{(\omega_4-\omega_3)S_3}{\hbar} + \mu^2(\dfrac{\omega_2}{\tau_{\mathrm{b}}} + \dfrac{\omega_4}{\tau_{\mathrm{a}}})} \quad (43)$$

It is clearly seen from equations (42) and (43) that both cooling load R and COP ε are functions of β_{h}' and β_{c}' for given β_{h}, β_{c}, β_0, q, a, c, λ, ω_1, ω_3, ω_{h}, μ and C_{e}. It is unable to evaluate the integral in the expression of cycle period time (equation (36)) in close form for the general case, therefore, it is unable to obtain the analytical fundamental relations between the optimal cooling load and COP. Using equations (42) and (43), one can plot three-dimensional diagrams of dimensionless cooling load ($R/R_{\max,\mu=0,C_{\mathrm{e}}=0}$, β_{h}', β_{c}') and COP (ε, β_{h}', β_{c}') for a set of given parameters as shown in Figures 2 and 3, where $R_{\max,\mu=0,C_{\mathrm{e}}=0}$ is the maximum cooling load for endoreversible case. For simplify, $\hbar = 1$ and $k_{\mathrm{B}} = 1$ are set in the following numerical calculations. According to Ref. [20] , the parameters used in numerical calculations are $\mathrm{a} = \mathrm{c} = 2$, $\mathrm{q} = \lambda = -0.5$, $\beta_{\mathrm{h}} = 0.5$, $\beta_{\mathrm{c}} = 1$, $\beta_0 = 1/1.8$, $\tau_{\mathrm{a}} = \tau_{\mathrm{b}} = 0.01$, $\omega_1 = 5$, $\omega_3 = 1$, $\omega_{\mathrm{c}} = 0.05$, $\mu = 0.01$ and $C_{\mathrm{e}} = 0.05$. Figure 2 shows that there exist optimal "temperatures" β_{h}' and β_{c}' of working medium in isothermal processes which lead to the maximum dimensionless cooling load for the spin-1/2 quantum Carnot refrigerator for given temperatures of hot and cold reservoirs and other parameters. As the result of effects of internal friction and heat leakage, the maximum dimensionless cooling load $(R/R_{\max,\mu=0,C_{\mathrm{e}}=0})_{\max} < 1$. From Figure 3, one can see clearly that there also exist optimal "temperatures" β_{h}' and β_{c}' for given temperatures of hot and cold reservoirs and other parameters which lead to the maximum COP when there exits a heat leakage, and the optimal "temperature" β_{h}' (or β_{c}') is close to the "temperature" of reservoirs β_{h} (or β_{c}).

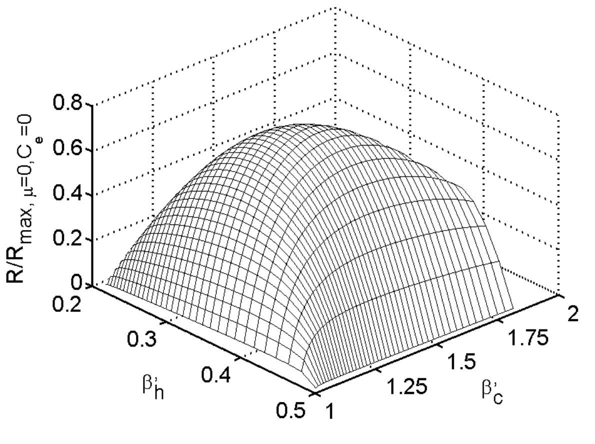

Figure 2. Dimensionless cooling load $R/R_{\max,\mu=0,C_{\mathrm{e}}=0}$ versus "temperatures" β_{h}' and β_{c}'

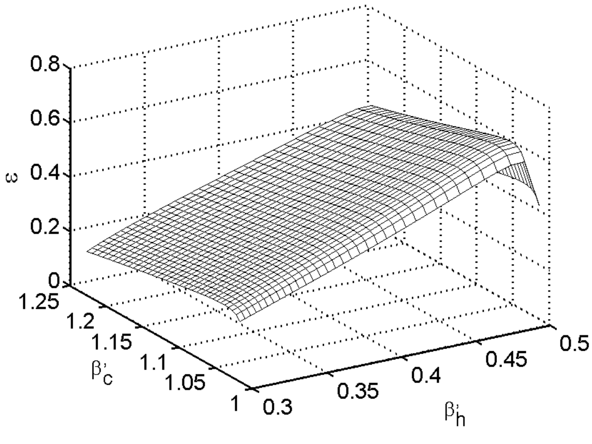

Figure 3. COP ε versus "temperatures" β_{h}' and β_{c}'

6. Cooling load and COP optimization at classical limit

When the temperatures of two heat reservoirs and working medium are high enough, i.e. $\beta\omega << 1$, the results obtained above can be simplified. At the first order approximation, equations (29), (30), (34) and (35) can be, respectively, simplified to

$$\omega_2 = \frac{\beta_{\mathrm{h}}'\omega_1\tau_{\mathrm{b}} - 4\mu^2}{\beta_{\mathrm{c}}'\tau_{\mathrm{b}}} \quad (44)$$

$$\omega_4 = \frac{\beta_c' \omega_3 \tau_a - 4\mu^2}{\beta_h' \tau_a} \tag{45}$$

$$\tau_h = \frac{1}{4a\hbar^2(\alpha_h - 1)} \ln\frac{\omega_1}{\omega_4} \tag{46}$$

$$\tau_c = \frac{1}{4a\hbar^2(\alpha_c - 1)} \ln\frac{\omega_3}{\omega_2} \tag{47}$$

With the help of equations (44)-(47), equations (20), (21), (36), (38) and (41)-(43) can be, respectively, simplified to

$$Q_h' = \frac{\omega_1^2 \beta_h'^2 \tau_a^2 - (\beta_c' \omega_3 \tau_a - 4\mu^2)^2}{8\beta_h' \tau_a^2} \tag{48}$$

$$Q_c' = \frac{(\beta_h' \omega_1 \tau_b - 4\mu^2)^2 - \beta_c'^2 \tau_b^2 \omega_3^2}{8\beta_c' \tau_b^2} \tag{49}$$

$$\tau = \frac{\begin{array}{c}\beta_h'(\beta_c - \beta_c')\ln[\beta_h' \tau_a \omega_1 / (\beta_c' \omega_3 \tau_a - 4\mu^2)] \\ + \beta_c'(\beta_h - \beta_h')\ln[\beta_c' \tau_b \omega_3 / (\beta_h' \omega_1 \tau_b - 4\mu^2)] + 4a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')(\tau_a + \tau_b)\end{array}}{4a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')} \tag{50}$$

$$\dot{Q}_e \approx C_e[2c\hbar\omega_c(1 + \lambda\hbar\beta_h\omega_c)/\beta_c](\beta_c - \beta_h) = C_e\alpha(\beta_c - \beta_h) \tag{51}$$

$$P_{in} = \frac{\begin{array}{c}a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')[\beta_h'^2 \beta_c' \tau_a^2 \tau_b^2 \omega_1^2 \\ + \beta_h' \beta_c'^2 \tau_a^2 \tau_b^2 \omega_3^2 - \beta_c' \tau_b^2 (\beta_c' \omega_3 \tau_a - 4\mu^2)^2 - \beta_h' \tau_a^2 (\beta_h' \omega_1 \tau_b - 4\mu^2)^2]\end{array}}{\begin{array}{c}2\beta_h' \beta_c' \tau_a^2 \tau_b^2 \{\beta_h'(\beta_c - \beta_c')\ln[\beta_h' \tau_a \omega_1 / (\beta_c' \omega_3 \tau_a - 4\mu^2)] + \beta_c'(\beta_h - \beta_h') \\ \times \ln[\beta_c' \tau_b \omega_3 / (\beta_h' \omega_1 \tau_b - 4\mu^2)] + 4a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')(\tau_a + \tau_b)\}\end{array}} \tag{52}$$

$$R = \frac{a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')[(\beta_h' \omega_1 \tau_b - 4\mu^2)^2 - \beta_c'^2 \tau_b^2 \omega_3^2]}{\begin{array}{c}2\beta_c' \tau_b^2 \{\beta_h'(\beta_c - \beta_c')\ln[\beta_h' \tau_a \omega_1 / (\beta_c' \omega_3 \tau_a - 4\mu^2)] + \beta_c'(\beta_h - \beta_h') \\ \times \ln[\beta_c' \tau_b \omega_3 / (\beta_h' \omega_1 \tau_b - 4\mu^2)] + 4a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')(\tau_a + \tau_b)\}\end{array}} - C_e\alpha(\beta_c - \beta_h) \tag{53}$$

$$\varepsilon = \frac{\beta_h' \tau_a^2 [(\beta_h' \omega_1 \tau_b - 4\mu^2)^2 - \beta_c'^2 \tau_b^2 \omega_3^2 - 8\beta_c' \tau_b^2 C_e \alpha(\beta_c - \beta_h)\tau]}{\beta_h'^2 \beta_c' \tau_a^2 \tau_b^2 \omega_1^2 + \beta_h' \beta_c'^2 \tau_a^2 \tau_b^2 \omega_3^2 - \beta_c' \tau_b^2 (\beta_c' \tau_a \omega_3 - 4\mu^2)^2 - \beta_h' \tau_a^2 (\beta_h' \tau_b \omega_1 - 4\mu^2)^2} \tag{54}$$

where $\alpha = 2c\hbar\omega_c(1 + \lambda\hbar\beta_h\omega_c)/\beta_c$.

Based on equations (53) and (54), it is still hard to optimize the cooling load and COP of the refrigerator and to obtain the fundamental optimal relations between the cooling load and COP analytically at high temperature limit. Therefore, the optimization problem is solved numerically in the following analysis. Using equations (53) and (54) , one can plot three-dimensional diagrams of dimensionless cooling load ($R/R_{max,\mu=0,C_e=0}$, β_h', β_c') and COP (ε, β_h', β_c') for a set of given parameters as shown in Figures 4 and 5, where $R_{max,\mu=0,C_e=0}$ is the maximum ecological function for endoreversible case at high temperature limit. For simplify, $\hbar = 1$ and $k_B = 1$ are set in the following numerical calculations. According to Ref. [20], the parameters used in numerical calculations are $a = c = 2$, $\lambda = -0.5$, $\beta_h = 1/300$, $\beta_c = 1/260$, $\beta_0 = 1/290$, $\tau_a = \tau_b = 0.01$, $\omega_1 = 12$, $\omega_3 = 2$, $\omega_c = 6$, $\mu = 0.001$ and $C_e = 0.0001$. Comparison between Figures 4 and 2 shows that the relationship among $R/R_{max,\mu=0,C_e=0}$ and β_h', β_c' at high temperature limit is similar to the relationship in general case, and there also exists a maximum dimensionless cooling load for the spin-1/2 quantum Carnot refrigerator. As the result of effects of internal friction and heat leakage, the maximum dimensionless cooling load $(R/R_{\max,\mu=0,C_e=0})_{\max} < 1$. Comparison between Figures 5 and 3 shows that the relationship among COP and β'_h, β'_c at high temperature limit is similar to the relationship in general case, and there also exist optimal "temperatures" β'_h and β'_c which lead to the maximum COP for the spin-1/2 quantum Carnot refrigerator for given temperatures of hot and cold reservoirs and other parameters when there exits a heat leakage, and the optimal "temperature" β'_h (or β'_c) is also close to the "temperature" of reservoirs β_h (or β_c).

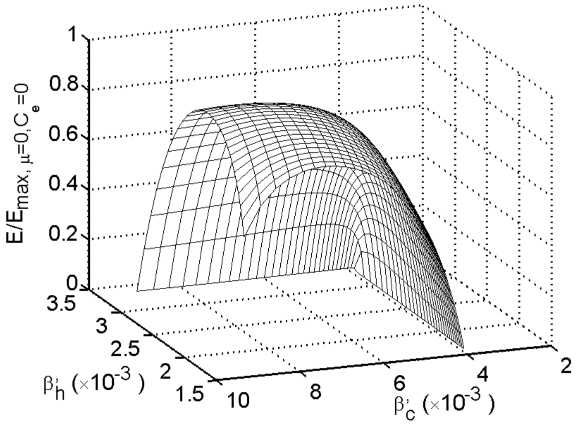

Figure 4. Dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus "temperatures" β'_h and β'_c at high temperature limit

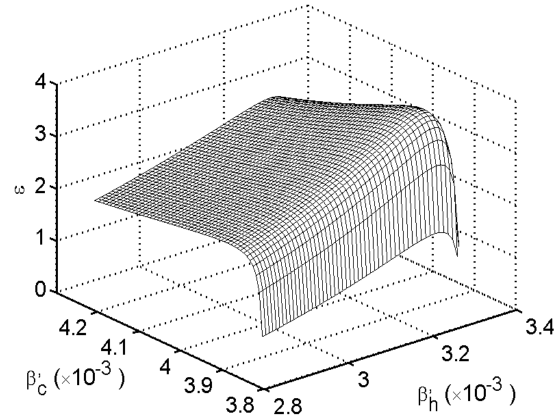

Figure 5. COP ε versus "temperatures" β'_h and β'_c at high temperature limit

In order to determine the optimal cooling load of the quantum refrigerator for a fixed COP or the optimal COP for a fixed cooling load, one can introduce Lagrangian functions $L_1 = R + \lambda_1\varepsilon$ and $L_2 = \varepsilon + \lambda_2 R$, where λ_1 and λ_2 are two Lagrangian multipliers. Theoretically, solving the Euler-Lagrange equations $\partial L_1/\partial\beta'_h = 0$, $\partial L_1/\partial\beta'_c = 0$ or $\partial L_2/\partial\beta'_h = 0$, $\partial L_2/\partial\beta'_c = 0$ gives the optimal relation between β'_h and β'_c. However, Combining equations (53) and (54) with the Euler-Lagrange equations above, one can find that it is hard to solve these equations analytically due to the strong complexity and nonlinearity. Therefore, the Euler-Lagrange equations are solved numerically in the following analysis. Figures 6 and 7 give the fundamental optimal relation between the dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ and COP ε. Except μ and C_e, the values of other parameters used in numerical calculations are the same as those used in Figure 4. From Figures 6 and 7, one can see clearly that the $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ curves are parabolic-like ones and the dimensionless cooling load has a maximum when there is no heat leakage $Q_e = 0$. The $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ curves are loop-shaped ones when there exists heat leakage $Q_e \neq 0$, the dimensionless cooling load has a maximum and the COP also has a maximum. The internal friction μ affects strongly both on dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ and COP ε, and both the dimensionless cooling load and COP decrease as the internal friction μ increases. For a fixed internal friction μ, both the dimensionless cooling load and COP decrease as the heat leakage increases. There are two different corresponding COPs for a given dimensionless cooling load (except the maximum dimensionless cooling load) and the refrigerator should work at the point that the COP is higher.

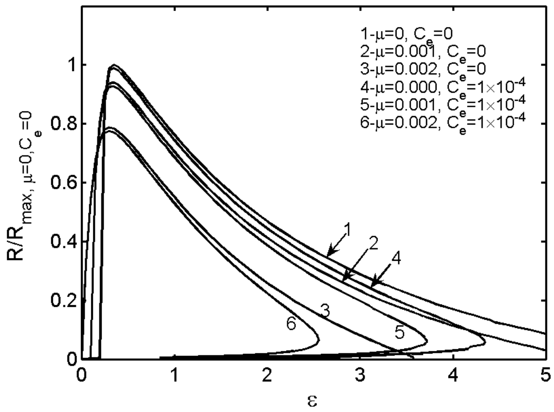

Figure 6. Effects of μ and $C_{\rm e}$ on dimensionless cooling load $R/R_{\max,\mu=0,C_{\rm e}=0}$ versus COP ε

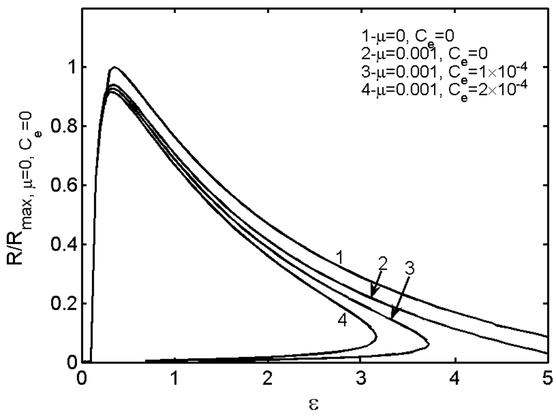

Figure 7. Effects of μ and $C_{\rm e}$ on dimensionless cooling load $R/R_{\max,\mu=0,C_{\rm e}=0}$ versus COP ε

7. Discussion

(1) If the cycle is an endoreversible one (i.e. $\mu=0$, $C_{\rm e}=0$), the time spent on the two adiabatic processes is negligible (i.e. $\tau_{\rm a}=\tau_{\rm b}=0$), and equations (36), (42) and (43) become

$$\tau=\frac{1}{2{\rm a}\hbar^2}\left[\int_{\beta'_{\rm h}\omega_4}^{\beta'_{\rm h}\omega_1}\frac{{\rm d}m_{\rm h}}{e^{{\rm q}\alpha_{\rm h}m_{\rm h}}(e^{\alpha_{\rm h}m_{\rm h}}-e^{m_{\rm h}})(1+e^{-m_{\rm h}})}+\int_{\beta'_{\rm c}\omega_2}^{\beta'_{\rm c}\omega_3}\frac{{\rm d}m_{\rm c}}{e^{{\rm q}\alpha_{\rm c}m_{\rm c}}(e^{\alpha_{\rm c}m_{\rm c}}-e^{m_{\rm c}})(1+e^{-m_{\rm c}})}\right] \tag{55}$$

$$R=\beta'_{\rm h}\left[\frac{S_3\omega_3}{\hbar\beta'_{\rm h}}-\frac{S_2\omega_1}{\hbar\beta'_{\rm c}}+\frac{1}{\beta'_{\rm h}\beta'_{\rm c}}\ln\frac{\cosh(\beta'_{\rm c}\omega_3/2)}{\cosh(\beta'_{\rm h}\omega_1/2)}\right]\tau^{-1} \tag{56}$$

$$\varepsilon=\frac{\beta'_{\rm h}}{\beta'_{\rm c}-\beta'_{\rm h}} \tag{57}$$

At high temperature limit, equations (55) and (56) can be simplified to

$$\tau=\frac{(\beta'_{\rm h}\beta_{\rm c}-\beta'_{\rm c}\beta_{\rm h})\ln[\beta'_{\rm h}\omega_1/(\beta'_{\rm c}\omega_3)]}{4{\rm a}\hbar^2(\beta_{\rm h}-\beta'_{\rm h})(\beta_{\rm c}-\beta'_{\rm c})} \tag{58}$$

$$R=\frac{{\rm a}\hbar^2(\beta_{\rm h}-\beta'_{\rm h})(\beta_{\rm c}-\beta'_{\rm c})(\beta'^2_{\rm h}\omega_1^2-\beta'^2_{\rm c}\omega_3^2)}{2\beta'_{\rm c}(\beta'_{\rm h}\beta_{\rm c}-\beta'_{\rm c}\beta_{\rm h})\ln[\beta'_{\rm h}\omega_1/(\beta'_{\rm c}\omega_3)]} \tag{59}$$

From equations (57) and (59), one can derive the fundamental optimal relation between cooling load and COP of the endoreversible quantum Carnot refrigerator analytically

$$R=\frac{{\rm a}\hbar^2[\varepsilon^2\omega_1^2-(1+\varepsilon)^2\omega_3^2][(1+\varepsilon)\beta_{\rm h}-\varepsilon\beta_{\rm c}]}{8\varepsilon(1+\varepsilon)^2\ln[\varepsilon\omega_1/(\omega_3+\omega_3\varepsilon)]} \tag{60}$$

(2) If there is no bypass heat leakage in the cycle (i.e. $C_{\rm e}=0$), equations (42) and (43) become

$$R=\left[\frac{S_3\omega_3-S_2\omega_2}{\hbar}+\frac{1}{\beta'_{\rm c}}\ln\frac{\cosh(\beta'_{\rm c}\omega_3/2)}{\cosh(\beta'_{\rm c}\omega_2/2)}\right]\tau^{-1} \tag{61}$$

$$\begin{aligned}\varepsilon=&\left[\frac{S_3\omega_3-S_2\omega_2}{\hbar}+\frac{1}{\beta'_{\rm c}}\ln\frac{\cosh(\beta'_{\rm c}\omega_3/2)}{\cosh(\beta'_{\rm c}\omega_2/2)}\right]\left[\frac{1}{\beta'_{\rm h}}\ln\frac{\cosh(\beta'_{\rm h}\omega_4/2)}{\cosh(\beta'_{\rm h}\omega_1/2)}\right.\\&\left.+\frac{1}{\beta'_{\rm c}}\ln\frac{\cosh(\beta'_{\rm c}\omega_2/2)}{\cosh(\beta'_{\rm c}\omega_3/2)}+\frac{(\omega_2-\omega_1)S_1}{\hbar}+\frac{(\omega_4-\omega_3)S_3}{\hbar}+\mu^2\left(\frac{\omega_2}{\tau_{\rm b}}+\frac{\omega_4}{\tau_{\rm a}}\right)\right]^{-1}\end{aligned} \tag{62}$$

The expression of cycle period of the irreversible quantum Carnot refrigerator with heat resistance and internal friction is still equation (36) due to the fact that the cycle period is independent of heat leakage. At high temperature limit, equations (61) and (62) can be simplified to

$$R = \frac{a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')[(\beta_h'\omega_1\tau_b - 4\mu^2)^2 - \beta_c'^2\tau_b^2\omega_3^2]}{2\beta_c'\tau_b^2[\beta_h'(\beta_c - \beta_c')\ln(\beta_h'\tau_a\omega_1/(\beta_c'\omega_3\tau_a - 4\mu^2)) + \beta_c'(\beta_h - \beta_h') \times\ln(\beta_c'\tau_b\omega_3/(\beta_h'\omega_1\tau_b - 4\mu^2)) + 4a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')(\tau_a + \tau_b)]} \tag{63}$$

$$\varepsilon = \frac{\beta_h'\tau_a^2[(\beta_h'\omega_1\tau_b - 4\mu^2)^2 - \beta_c'^2\tau_b^2\omega_3^2]}{\beta_h'^2\beta_c'\tau_a^2\tau_b^2\omega_1^2 + \beta_h'\beta_c'^2\tau_a^2\tau_b^2\omega_3^2 - \beta_c'\tau_b^2(\beta_c'\tau_a\omega_3 - 4\mu^2)^2 - \beta_h'\tau_a^2(\beta_h'\tau_b\omega_1 - 4\mu^2)^2} \tag{64}$$

Based on equations (63) and (64), it is hard to optimize cooling load and COP of the refrigerator and to obtain the fundamental relations between the optimal cooling load and COP analytically. Fig. 6 gives the characteristic curves of dimensionless cooling load $R/R_{\max,\mu=0,C_e=0}$ versus COP ε of the quantum refrigerator when there is no heat leakage in the cycle by numerical calculation, and the $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ curves are parabolic-like ones and the dimensionless cooling load has a maximum.

(3) If there is no internal friction in the cycle (i.e. $\mu = 0$), the time spent on the two adiabatic processes is negligible (i.e. $\tau_a = \tau_b = 0$), and equations (42) and (43) become

$$R = \beta_h'[\frac{S_3\omega_3}{\hbar\beta_h'} - \frac{S_2\omega_1}{\hbar\beta_c'} + \frac{1}{\beta_h'\beta_c'}\ln\frac{\cosh(\beta_c'\omega_3/2)}{\cosh(\beta_h'\omega_1/2)}]\tau^{-1} - 2C_e c\hbar\omega_c e^{\lambda\hbar\beta_h\omega_c}[1-(e^{\hbar\beta_h\omega_c} - 1)n_c] \tag{65}$$

$$\varepsilon = \frac{\beta_h'}{\beta_c' - \beta_h'} - \frac{2C_e c\hbar\omega_c e^{\lambda\hbar\beta_h\omega_c}[1-(e^{\hbar\beta_h\omega_c} - 1)n_c]\tau}{(\beta_c' - \beta_h')[\frac{S_3\omega_3}{\hbar\beta_h'} - \frac{S_1\omega_1}{\hbar\beta_c'} + \frac{1}{\beta_h'\beta_c'}\ln\frac{\cosh(\beta_c'\omega_3/2)}{\cosh(\beta_h'\omega_1/2)}]} \tag{66}$$

The expression of cycle period of the irreversible quantum Carnot refrigerator with heat resistance and heat leakage still is equation (55) due to the fact that the cycle period is independent of heat leakage. At high temperature limit, equations (65) and (66) can be simplified to

$$R = \frac{a\hbar^2(\beta_h - \beta_h')(\beta_c - \beta_c')(\beta_h'^2\omega_1^2 - \beta_c'^2\omega_3^2)}{2\beta_c'(\beta_h'\beta_c - \beta_c'\beta_h)\ln[\beta_h'\omega_1/(\beta_c'\omega_3)]} - C_e\alpha(\beta_c - \beta_h) \tag{67}$$

$$\varepsilon = \frac{\beta_h'\tau_a^2[(\beta_h'\omega_1\tau_b - 4\mu^2)^2 - \beta_c'^2\tau_b^2\omega_3^2 - 8\beta_c'\tau_b^2C_e\alpha(\beta_c - \beta_h)\tau]}{\beta_h'^2\beta_c'\tau_a^2\tau_b^2\omega_1^2 + \beta_h'\beta_c'^2\tau_a^2\tau_b^2\omega_3^2 - \beta_c'\tau_b^2(\beta_c'\tau_a\omega_3 - 4\mu^2)^2 - \beta_h'\tau_a^2(\beta_h'\tau_b\omega_1 - 4\mu^2)^2} \tag{68}$$

From equations (58), (67) and (68), for given S_1 and S_3, one can drive the maximum cooling load and corresponding COP of the irreversible quantum Carnot refrigerator with heat resistance and heat leakage analytically.

8. Conclusion

A model of an irreversible quantum Carnot refrigerator using non-interacting spin-1/2 systems as working medium is established in this paper, and the irreversibilities of heat resistance, internal friction and bypass heat leakage are considered. The refrigeration cycle is consisting of two isothermal branches and two irreversible adiabatic branches. This paper gives expressions of some important performance parameters, such as cycle period, cooling load and COP for the irreversible quantum Carnot refrigerator using the quantum master equation, semi-group approach and finite time thermodynamics. The optimal performance of the refrigerator at high temperature limit is analyzed in detail with numerical examples, the optimal characteristic curves of cooling load versus COP are plotted, effects of internal friction and heat leakage one the optimal performance are discussed. Both the cooling load and COP have maximum. At high temperature limit, the cooling load versus COP curves $R/R_{\max,\mu=0,C_e=0} - \varepsilon$ are parabolic-like ones when there is no heat leakage, and the cooling load has a maximum. The cooling load versus COP curves

$R/R_{\max,\mu=0,C_e=0}-\varepsilon$ are loop-shaped ones when there exists heat leakage, and both the cooling load and COP have maximums. The internal friction does decrease the cooling load and COP, but not change the shape of the $R/R_{\max,\mu=0,C_e=0}-\varepsilon$ curves. The obtained results can offer further understanding of the optimal performance of the irreversible quantum Carnot refrigerator with spin-1/2 systems working medium, as well as the similarities and differences between quantum thermodynamic cycle and the cycles working with classical working medium. They can provide some theoretical guidelines for optimal design and selection of operational parameters of real quantum refrigerators.

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Nomenclature

a	parameter of heat reservoir (s^{-1})
B	heat reservoir
$\vec{B}$	external magnetic field (T)
$\hat{B}^{+}$, $\hat{B}^{-}$	creation and annihilation operators of thermal phonons of reservoir
C_e	dimensionless factor connected with heat leakage
c	parameter of heat reservoir (s^{-1})
E	internal energy of the spin-1/2 systems (J)
$\hat{H}$	Hamiltonian
$\hbar$	reduced Planck's constant ($J\cdot s$)
k_B	Boltzmann constant (J/K)
L_1, L_2	Lagrangian functions
$\hat{M}$	magnetic moment operator
m	intermediate variable
n_c	population of the thermal phonons of the cold reservoir
P_{in}	power input
Q	amount of heat exchange (J)
$\hat{Q}_\alpha$, $\hat{Q}_\alpha^{+}$	operator in the Hilbert space of the system and Hermitian conjugate
Q'	amount of heat exchange between heat reservoir and working medium (J)
$\dot{Q}$	rate of heat flow (W)
q	parameter of heat reservoir
R	cooling load (W)
S	expectation value of $\hat{S}_z$
$\hat{S}_{+}$, $\hat{S}_{-}$	spin creation and annihilation operators
$\hat{S}(\hat{S}_x,\hat{S}_y,\hat{S}_z)$	spin operator
S_{eq}	asymptotic value of S
T	absolute temperature (K)
T'	absolute temperature of the working medium (K)
t	time (s)
W	work (W)

Greek symbols

α	intermediate variable
β	"temperature" $\beta=1/(k_B T)$ (J^{-1})
β'	"temperature" of working medium $\beta'=1/(k_B T')$ (J^{-1})
γ_{+}, γ_{-}	phenomenological positive coefficients
ε	coefficient of performance
λ	parameter of the heat reservoir
λ_1, λ_2	Lagrangian multipliers
μ	friction coefficient
μ_B	Bohr magneton (J/T)
$\hat{\sigma}(\hat{\sigma}_x,\hat{\sigma}_y,\hat{\sigma}_z)$	Pauli operator
σ	entropy generation rate (W/K)
$\hat{\Gamma}$	interaction strength operator
τ	time (s) / cycle period (s)
ω	frequency of the thermal phonons (s^{-1})

Subscripts

B	heat reservoir
c	cold side
h	hot side
S	working medium system
SB	interaction between heat reservoir and working medium system
$\mu=0,C_e=0$	maximum point for endoreversible case
0	environment
1, 2, 3, 4	cycle states

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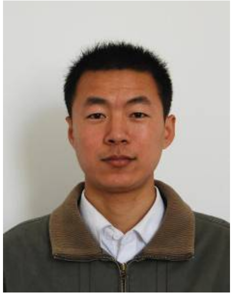

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