Effect of heat transfer law on the finite-time exergoeconomic performance of a generalized irreversible Carnot heat engine

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Abstract
The analytical expression for profit rate of a generalized irreversible Carnot heat engine cycle based on a generalized radiative heat transfer law \( q \propto \Delta(T^n) \) is derived by applying the finite time exergoeconomic method, taking into account several additional irreversibilities, such as heat resistance, heat leakage and other undesirable irreversible factors. The compromise optimization between economics (profit rate) and the efficiency was obtained by searching the efficiency at maximum profit rate, which is termed as the finite time exergoeconomic performance bound.

Keywords: Finite-time thermodynamics; Generalized irreversible Carnot heat engine; Exergoeconomic performance; Generalized thermodynamic optimization; Heat transfer law.

1. Introduction
Recently, the intensive consumption of energy and the exhaustion of resources lead to the rising costs for energy. Hence, from the economic perspective, improvement of engine performance is urgently required. Finite-time thermodynamics [1-8] is a powerful tool often used to optimize thermodynamic parameters including power, efficiency, entropy generation, effectiveness, cooling load, heating load, loss of exergy, etc.

Nowadays, systems like heat engines are analyzed and designed based on the consideration of both thermodynamic parameters and cost accounting requirements after the research of Salamon and Nitzan [9, 10], which was to maximize the profit of an endoreversible heat engine by a combination of a thermodynamic analysis with an engineer economic analysis. In order to distinguish this method from the endoreversible analysis optimizing pure thermodynamic objectives, Chen et al. [11-17] analyzed the profit rate of thermal systems by attributing costs to input and output exergy and termed this method as finite-time exergoeconomic analysis and its performance bound at maximum profit as finite-time exergoeconomic performance bound. Other researches seeking for best economic performance of thermal systems were carried out on endoreversible engines, refrigerators and heat pumps by Ibrahim et al. [18], De Vos [19, 20] and Bejan [21], with the only irreversibility restricted to the heat transfer between the working fluid and the heat reservoirs. De Vos [19, 20] applied the Newton (linear) heat transfer law to
derive the relation between the optimal efficiency and economic returns when carrying out
thermoeconomics analysis for heat engine. Chen et al. [22] investigated the endoreversible
thermoeconomic performance of heat engine with the heat transfer between the working fluid and the
heat reservoirs obeying linear phenomenological law. Sahin et al. [23-26] proposed an optimization
criterion considering thermodynamic parameters per unit total cost.
In many pioneer works concerning finite-time exergoeconomic optimization for heat engines, the basic
thermodynamic model is endoreversible. However, a real heat engine will operate in an irreversible
power cycle which incorporates several internal and external irreversibilities, such as heat resistance,
bypass heat leakage, friction, turbulence and other undesirable irreversibility factors. Considering
external and internal irreversibilities, Chen et al. [16-27] established a generalized irreversible Carnot
heat engine model. As heat transfer is not necessarily Newtonian or linear phenomenological, a further
step made in this paper is to establish a fundamental optimal relationship between profit and efficiency of
the generalized irreversible Carnot heat engine based on generalized radiative heat transfer
law \( q \propto \Delta(T^n) \). The result obtained by searching the optimum efficiency at maximum profit involved
three common heat transfer laws: Newton’s law \((n = 1)\), the linear phenomenological law \((n = -1)\), and
the radiative heat transfer law \((n = 4)\). The relative studies can be seen in Refs. [28-35].

2. Cycle model and performance analysis
In order to conduct the simulation closer to the performance of an actual heat engine, Chen, et al. [16,
27] established a generalized irreversible steady flow Carnot heat engine cycle model as shown in Figure
1, considering heat resistance, heat leakage, and internal irreversibilities. The working fluid in this
generalized irreversible engine with constant-temperature heat-reservoirs flows steadily. The system
undergoes a cycle which consists of four irreversible processes, two isothermal and two adiabatic.
External irreversibilities are caused by the heat resistance existed in the high- and low-temperature heat-
exchangers. Heat-transfer between the heat engine and its surrounding heat reservoirs leads to the
difference between the working fluid temperature \((T_{HC} \text{ and } T_{LC})\) and the heat-reservoir temperature
\((T_H \text{ and } T_L)\). These temperatures are related to one another in the following order:

\[ T_H > T_{HC} > T_{LC} > T_L \tag{1} \]

A constant rate of heat leakage \((q)\) from the heat source at the temperature \(T_H\) to the heat sink at \(T_L\) is
assumed for this system, which yields,

\[ Q_H = Q_{HC} + q \tag{2} \]

\[ Q_L = Q_{LC} + q \tag{3} \]

where \(Q_{HC}\) and \(Q_{LC}\) are the rates of heat-transfer supplied by the heat source and released to the heat
sink by the working fluid, respectively; \(Q_H\) and \(Q_L\) are the real rates of heat-supply and heat-release,
respectively. Assuming the heat-transfer law obeys \(q \propto \Delta(T^n)\), the rate of heat leakage can be
expressed as

\[ q = C_i(T_H^n - T_L^n) \tag{4} \]

where \(C_i\) is the heat leakage coefficient.
When analyzing actual heat engines, heat resistance and heat leakage discussed above are not the only
irreversibilities. Irreversibilities caused by friction, turbulence, and non-equilibrium activities inside the
working fluid are also required to be considered. Thus when compared with an endoreversible Carnot
heat engine of the same heat input, the generalized irreversible Carnot engine can deliver less power and
release more heat to the heat sink. Hence, the rate of heat flow \((Q_{LC})\) to the heat sink for the generalized
irreversible Carnot engine is larger than that \((Q'_{LC})\) for the endoreversible Carnot engine with the same input. A constant coefficient \((\varphi)\) is introduced in the following expression to generally characterize the additional miscellaneous irreversible effects

\[
\varphi = \frac{Q_{LC}}{Q_{LC}} \geq 1
\]  

(5)

Application of the second law of thermodynamics yields,

\[
\frac{Q_{LC}}{T_{LC}} = \frac{Q_{HC}}{T_{HC}}
\]  

(6)

Combining Eqs. (5) and (6) gives

\[
Q_{LC} = \varphi \frac{Q_{HC}}{x}
\]  

(7)

where \(x = \frac{T_{HC}}{T_{LC}}\) \((1 \leq x \leq T_H / T_L)\) is the temperature ratio of the working fluid.

Application of the first law of thermodynamics gives the expressions of power output and thermal efficiency, respectively

\[
P = Q_H - Q_L = Q_{HC} - Q_{LC}
\]  

(8)

\[
\eta = \frac{P}{Q_H} = \frac{Q_{HC} - Q_{LC}}{(Q_{HC} + q)}
\]  

(9)

\[\text{Figure 1. The generalized irreversible Carnot heat engine cycle model}\]

Assuming the rates of the heat flow in the heat-exchangers follow the generalized radiative heat transfer law, \(q \propto \Delta(T^n)\), where \(n\) is a heat transfer exponent, with \(n = 1\) representing the Newton’s law, \(n = -1\) representing the linear phenomenological law and \(n = 4\) representing the radiative heat transfer law. Then
\[ Q_{HC} = k_1 F_1 (T_H^n - T_{HC}^n) \]  
\[ Q_{LC} = k_2 F_2 (T_L^n - T_L^n) \]  
where \( k_1 \) and \( k_2 \) are the overall heat-transfer coefficients of high- and low-temperature side heat-exchangers, \( F_1 \) and \( F_2 \) are the surface areas of high- and low-temperature side heat-exchangers. The total heat transfer surface area of the two heat exchangers is taken as a constant, that is
\[ F_1 + F_2 = F_T \]  
And a ratio \( (f) \) of heat exchanger area is defined as
\[ f = F_1 / F_2 \]  
Assuming that the prices of the work output and exergy input are \( \psi_1 \) and \( \psi_2 \), respectively, the profit rate (profit per unit time) of the generalized irreversible Carnot heat engine is [11]
\[ \pi = \psi_1 P - \psi_2 A \]  
where \( A \) is the rate of exergy input of the heat engine which can be expressed as
\[ A = Q_H (1 - T_0 / T_H) - Q_L (1 - T_0 / T_L) = Q_H \varepsilon_1 - Q_L \varepsilon_2 \]  
where \( \varepsilon_i \) is the Carnot coefficient of the reservoir and \( T_0 \) is the environmental temperature. Combining Eqs. (2)-(3) and (7)-(15) gives
\[ \eta = B' (x - \phi) [T_H^n - (x T_L^n)] / \{ B' x [T_H^n - (x T_L^n)] + q \} \]  
\[ \pi = B' \psi_1 [T_H^n - (x T_L^n)] [x (1 - \varepsilon_1 \psi_2 / \psi_1) - \phi (1 - \varepsilon_2 \psi_2 / \psi_1)] + q \psi_2 (\varepsilon_2 - \varepsilon_1) \]  
where \( B' = k_1 F_T / [(1 + f) (x + x^n \phi f k_1 / k_2)] \).
Maximizing \( \eta \) and \( \pi \) with respect to \( f \) by setting \( d\eta / df = 0 \) and \( d\pi / df = 0 \) using Eqs. (16) and (17) yields the same optimal ratio of heat-exchanger area \( (f_{opt}) \)
\[ f = f_{opt} = [k_2 / (x^{n-1} \phi k_1)]^{0.5} \]  
Substituting Eq. (18) into Eqs. (16) and (17), respectively, yields the optimal efficiency and profit rate in the following forms:
\[ \eta = B (x - \phi) [T_H^n - (x T_L^n)] / \{ B x [T_H^n - (x T_L^n)] + q \} \]  
\[ \pi = B \psi_1 [T_H^n - (x T_L^n)] [x (1 - \varepsilon_1 \psi_2 / \psi_1) - \phi (1 - \varepsilon_2 \psi_2 / \psi_1)] + q \psi_2 (\varepsilon_2 - \varepsilon_1) \]  
where \( B = k_1 F_T / [(x^{0.5} + (x^n \phi k_1 / k_2)^{0.5})^2] \).
Maximum profit rate and maximum efficiency with respect to temperature ratio can be derived by taking derivatives of Eq.(19) and Eq.(20) with respect to \( x \). However, in general, the optimal temperature ratio
\( x_\pi \) at maximum profit rate \( \pi_{\text{max}} \) does not equal to the optimal temperature ratio \( \eta_{\text{max}} \). The optimal temperature ratio \( x_\pi \) at maximum profit rate \( \pi_{\text{max}} \) can be derived by taking the derivative of the profit rate with temperature ratio and setting it equal to zero \( (d\pi/dx = 0) \). By substituting the optimal temperature ratio \( x_\pi \) into Eq. (20), the maximum profit rate can be achieved. Furthermore, the finite-time exergoeconomic bound of the generalized irreversible Carnot heat engine will be obtained by substituting the optimal temperature ratio \( x_\pi \) with respect to maximum profit rate into Eq. (19).

3. Discussions

3.1 Effects of various losses on the performance

If \( \phi = 1 \) and \( q > 0 \), Eqs. (19) and (20) become

\[
\eta = B_{en}(x-1)[T_H^n - (xT_L)^n]/\{B_{en}(T_H^n - (xT_L)^n) + q\} \tag{21}
\]

\[
\pi = B_{en}\psi_1[T_H^n - (xT_L)^n][x(1-\epsilon_1\psi_2/\psi_1) - (1-\epsilon_2\psi_2/\psi_1)] + q\psi_2(\epsilon_2 - \epsilon_1) \tag{22}
\]

where \( B_{en} = k_1F_r/\lbrack x^{0.5} + (x^n k_1/k_2)^{0.5} \rbrack^2 \). Eqs. (21) and (22) are the relations between profit rate and efficiency of the irreversible Carnot heat engine with heat resistance and heat leakage losses.

If \( \phi > 1 \) and \( q = 0 \), Combining Eq. (19) and Eq. (20) gives

\[
\pi = B_{en}\phi\psi_1[T_H^n - (1-\eta)^n]\lbrack(1-\epsilon_1\psi_2/\psi_1)/(1-\eta) - (1-\epsilon_2\psi_2/\psi_1)\rbrack \tag{23}
\]

Eq. (23) is the relation between profit rate and efficiency of the irreversible Carnot heat engine with heat resistance and internal irreversibility losses.

If \( \phi = 1 \) and \( q = 0 \), Eq. (23) is reduced to

\[
\pi = B_{en}\psi_1[T_H^n - (1-\eta)^n]\lbrack(1-\epsilon_1\psi_2/\psi_1)/(1-\eta) - (1-\epsilon_2\psi_2/\psi_1)\rbrack \tag{24}
\]

Eq. (24) is the relation between profit rate and efficiency of the endoreversible Carnot heat engine [11].

3.2 Special cases

(1) Case of \( n = 1 \)

In the case of \( n = 1 \), Eq. (19) and Eq. (20) become:

\[
\eta = (x-\phi)(T_Hx^{-1}-T_L)/(T_H - xT_L + qB^{-1}) \tag{25}
\]

\[
\pi = B\psi_1(T_Hx^{-1}-T_L)[x(1-\epsilon_1\psi_2/\psi_1) - \psi(1-\epsilon_2\psi_2/\psi_1)] + q\psi_2(\epsilon_2 - \epsilon_1) \tag{26}
\]

where \( B = k_1F_r/[1 + (\phi k_1/k_2)^{0.5}]^2 \). Maximizing \( \pi \) with respect to \( x \) by setting \( d\pi/dx = 0 \) in Eq. (26) yields the optimal temperature ratio and the maximum profit rate of the heat engine:

\[
x_{\text{opt}} = (\phi - \epsilon_2\psi_2/\psi_1)^{0.5}/(T_L/\psi_1 - \epsilon_1\psi_2) \tag{27}
\]

\[
\pi_{\text{max}} = B\lbrack(T_H(\psi_1 - \epsilon_1\psi_2))^{0.5} - \phi(T_L(\psi_1 - \epsilon_2\psi_2))^{0.5}\rbrack^2 + \psi_2q(\epsilon_2 - \epsilon_1) \tag{28}
\]
Substituting Eq. (27) into Eq. (25) gives $\eta_z$, which is the finite-time exergoeconomic bound of generalized irreversible Carnot heat engine with Newton’s heat transfer law

$$
\eta_z = \frac{T_H + \phi T_L - (\phi T_H T_L)^{0.5} \left[ \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) / \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) \right]^{0.5} + \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) / \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) \right]^{0.5} + qB^\phi}{T_H - (\phi T_H T_L)^{0.5} \left[ \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) / \left( \psi_1 - \epsilon \psi_2 / \psi_1 \right) \right]^{0.5} + qB^\phi}
$$

(29)

(2) Case of $n = -1$

In the case of $n = -1$, Eq.(19) and Eq.(20) become

$$
\eta = B_1(x - \phi)(T_L^{-1} - xT_H^{-1})/\left[ B_1(x(T_L^{-1} - xT_H^{-1}) + q \right]
$$

(30)

$$
\pi = B_1 \psi_1 \left[ T_L^{-1} - xT_H^{-1} \right] \left[ \left( 1 - \epsilon \psi_2 / \psi_1 \right) - \phi(1 - \epsilon \psi_2 / \psi_1) \right] + q \psi_2 (\epsilon_2 - \epsilon_1)
$$

(31)

where $B_1 = k_i F_i / \left[ x + (\phi k_i / k_2)^{0.5} \right]^2$. Maximizing $\pi$ with respect to $x$ by setting $d\pi / dx = 0$ in Eq. (29) yields the optimal temperature ratio and the maximum profit rate of the heat engine

$$
x_{\text{opt}} = \frac{2T_H \phi(1 - \epsilon \psi_2 / \psi_1) + (\phi k_i / k_2)^{0.5} \left[ T_H \phi(1 - \epsilon \psi_2 / \psi_1) + T_H (1 - \epsilon \psi_2 / \psi_1) \right]}{2T_L \left( \phi k_i / k_2 \right)^{0.5} \left( 1 - \epsilon \psi_2 / \psi_1 \right) + T_L \phi(1 - \epsilon \psi_2 / \psi_1) + T_H (1 - \epsilon \psi_2 / \psi_1)}
$$

(32)

$$
\pi_{\text{max}} = B_1 \psi_1 \left[ T_L^{-1} - x_{\text{opt}} T_H^{-1} \right] \left[ \left( 1 - \epsilon \psi_2 / \psi_1 \right) - \phi(1 - \epsilon \psi_2 / \psi_1) \right] + q \psi_2 (\epsilon_2 - \epsilon_1)
$$

(33)

Substituting Eq. (32) into Eq. (30) gives $\eta_z$, which is the finite-time exergoeconomic bound of generalized irreversible Carnot heat engine based on linear phenomenological heat transfer law

$$
\eta_z = (x_{\text{opt}} - \phi) / \left[ x_{\text{opt}} + qT_H T_L / \left[ B_x (T_H - x_{\text{opt}} T_L) \right] \right]
$$

(34)

where $B_x = k_i F_i / \left[ x_{\text{opt}} + (\phi k_i / k_2)^{0.5} \right]^2$.

(3) Case of $n = 4$

In the case of $n = 4$, Eq.(19) and Eq.(20) become

$$
\eta = B_4(x - \phi)[(T_H^4 - xT_L^4) / \left[ B_4x(T_H^4 - xT_L^4) \right] + q]
$$

(35)

$$
\pi = B_4 \psi_1 \left[ T_H^4 - xT_L^4 \right] \left[ \left( 1 - \epsilon \psi_2 / \psi_1 \right) - \phi(1 - \epsilon \psi_2 / \psi_1) \right] + q \psi_2 (\epsilon_2 - \epsilon_1)
$$

(36)

where $B_4 = k_i F_i / \left[ x^{0.5} + x^2 (\phi k_i / k_2)^{0.5} \right]^2$. Eqs (35) and (36) are the relations between profit rate and efficiency of the irreversible Carnot heat engine based on the radiative heat transfer law.

The relationships between the profit rate and efficiency of the irreversible Carnot heat engine for all discussed cases are shown in Figure 2 and Figure 3. It can be concluded from the figure that the profit rate versus efficiency is a loop-shaped curve for all cases with heat leakage. For the cases without heat leakage, the profit rate decreases when the irreversibility factor $\phi$ increases with the shape of the curve remaining parabolic. For the case of $n < 0$, the optimal efficiency at the maximum profit rate increases with the increase of $n$ as shown in Figure 2. While, for the case of $n > 0$, the optimal efficiency at the maximum profit rate decreases with the increase of $n$ as shown in Figure 3. When $n$ increases, the influence of temperature on power becomes more remarkable. Hence, when $n$ is relatively large, by slightly sacrificing the efficiency, a significant increase of power can be achieved.
Figure 2. The influences of heat leak, internal irreversibility and heat transfer law on $\pi - \eta$ characteristic for $n < 0$

Figure 3. The influences of heat leak, internal irreversibility and heat transfer law on $\pi - \eta$ characteristic for $n > 0$

3.3 The effect of price ratio $\psi_2/\psi_1$

The finite-time exergoeconomic performance bound at the maximum profit rate is different from the classical reversible bound and the finite-time thermodynamic bound. It is dependent on $T_H$, $T_L$, $T_0$ and $\psi_2/\psi_1$. In order to ensure the process being potential profitable, $0 < \psi_2/\psi_1 < 1$ is required.

As the price of work output becomes very large compared with that of exergy input, i.e., $\psi_2/\psi_1 \to 0$, the function of the profit rate becomes
The optimization of the profit rate also leads to the maximization of the power output $P$ of the generalized irreversible heat engine cycle.

On the other hand, with the price of work output approaching the price of the exergy input, i.e. $\psi_2/\psi_1 \to 1$, the function of the profit rate becomes

$$\pi = \psi_1 (x - \varphi) [T_H^n - (xT_L^n)] = \psi_1 (Q_{HC} - Q_{LC}) = \psi_1 P$$

(37)

where $\sigma$ is the rate of entropy production of the generalized irreversible heat engine cycle. When maximizing the profit under this condition, minimization of the losses of exergy can be achieved. Eq. (38) indicates that the heat engine is always operating at a loss, unless it operates reversibly to reach the break-even point. Therefore, for any intermediate values of $\psi_2/\psi_1$, the finite-time exergoeconomic performance bound ($\eta_x$) lies between the finite-time thermodynamic performance bound and the reversible performance bound.

4. Conclusion

The relationship between the optimal profit rate and efficiency of a generalized irreversible Carnot heat engine is derived based on generalized radiative heat transfer law. The influence of different heat transfer laws and irreversibilities on this relationship has been discussed. The results are helpful for establishing a link among finite-time exergoeconomic performance bound, finite-time thermodynamic performance bound and the reversible performance bound.

Acknowledgments

This paper is supported by National Natural Science Foundation of China (Project No. 10905093).

References


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