Optimal configuration for a finite low-temperature source refrigerator cycle with heat transfer law $Q \propto (\Delta T^n)^m$

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Abstract
The optimal configuration of a refrigeration cycle operating between a finite low-temperature source and an infinite high-temperature sink are derived by using finite time thermodynamics based on a complex heat transfer law, including Newtonian heat transfer law, linear phenomenological heat transfer law, radiative heat transfer law, Dulong-Petit heat transfer law, generalized convective heat transfer law and generalized radiative heat transfer law, $Q \propto (\Delta T^n)^m$. In the refrigeration cycle model the only irreversibility of finite rate heat transfer is considered. The optimal relation between cooling load and coefficient of performance (COP) of the refrigeration cycle is also derived by using an equivalent temperature of low-temperature source. The obtained results include those with various heat transfer laws and infinite low-temperature source, and can provide some theoretical guidelines for the designs of practical refrigerators.

Keywords: Finite time thermodynamics; Finite heat capacity reservoir; Refrigeration cycle; Optimal configuration; Optimal performance; Heat transfer law.

1. Introduction
There are two standard problems in finite time thermodynamics [1-20], one is to determine the objective function limits and the relations between objective functions for the given thermodynamic system, and another is to determine the optimal thermodynamic process for the given optimization objectives (i.e. to determine the optimal configurations) for the system which serves as a model for real processes.

It is often the case in practice that the cooling load is generated from heat source which is carried by a finite amount of materiel with finite heat capacity, rather than from heat extracted from an isothermal, infinite spource. In the reversible (infinite-time) limit, the cycle, which extracts the maximum work from a finite heat source, is qualitatively different from the Carnot cycle, and its theoretical efficiency is considerably smaller [21].

The optimal configurations of heat engines under different given conditions were obtained by Ondrechen et al. [22], Yan and Chen [23, 24], Xiong et al. [25], Chen et al. [26-29] and Li et al. [30]. Chen [31] discussed the optimal configuration of a class of endoreversible refrigerators, for which only the irreversible heat transfer process is concerned. They derived that the endoreversible Carnot refrigerator is the optimal configuration of these endoreversible refrigerators with Newtonian heat transfer law.
according to the maximum coefficient of performance (COP) as the operating goal. Chen et al. [32] investigated the effect of heat leakage on the optimal configuration of refrigerator with consideration of finite heat capacity low-temperature source, infinite heat capacity high-temperature sink and Newtonian heat transfer law. Chen et al. [33] also given the unified description of endoreversible cycles for linear phenomenological heat transfer law \( Q \propto (\Delta T^{-1}) \). The results obtained in Refs. [24-27, 30] show that heat transfer law has the significant influences on the optimal configurations and performance of heat engine cycles, and a study on the effect of heat transfer law on optimal configuration of refrigeration cycles is necessary.

This paper will extend the previous work by using a complex heat transfer law, including Newtonian heat transfer law, linear phenomenological heat transfer law, radiative heat transfer law, Dulong-Petit heat transfer law, generalized convective heat transfer law and generalized radiative heat transfer law, \( Q \propto (\Delta T^r)^n \), in the heat transfer processes between the refrigerator and its surroundings, to find the optimal configuration of the variable-temperature heat-reservoir refrigeration cycles. In the refrigeration cycle model the only irreversibility of finite rate heat transfer is considered. The optimal relation between cooling load and COP of the refrigerator is also derived by using an equivalent temperature of low-temperature source. The optimal performance of endoreversible and irreversible Carnot refrigerator with infinite thermal-capacity (constant-temperature) heat reservoirs with the same heat transfer law were derived by Li et al. [34, 35]. The obtained results of this paper include those with various heat transfer laws and can provide some theoretical guidelines for the designs of practical refrigerators.

2. Refrigeration model

The generalized refrigeration cycle model and its surroundings to be considered in this paper are shown in Figure 1. The following assumptions are made for this model. The system adopted is a working fluid alternately connected to a heat source with finite heat capacity and to a heat sink with infinite heat capacity. The refrigerator operates in a cyclic fashion with a fixed time \( \tau \) allotted for each cycle. The low-temperature heat-source is assumed to have a constant heat-capacity \( C \), its temperature is given by \( T_x(t) \), and its initial temperature is given by \( T_x \). The high-temperature heat-sink is assumed, for simplicity, to be infinite in size and therefore it has a fixed temperature, \( T_H \). The heat transfer between heat source, heat sink and working fluid obey a complex law, including Newtonian heat transfer law, linear phenomenological heat transfer law, radiative heat transfer law, Dulong-Petit heat transfer law, generalized convective heat transfer law and generalized radiative heat transfer law, \( Q \propto (\Delta T^r)^n \). The absorbed and released heats of the working fluid are \( Q_L \) and \( Q_H \), respectively.

The two steps in the cycle during which the working fluid is disconnected from one reservoir and connected to another are taken to be reversibly adiabatic. It is assumed that these steps occur instantaneously, which implies that the temperature of the working fluid changes discontinuously.

![Figure 1. Model of the refrigeration cycle](image-url)
3. Optimal configuration

Considering that the heat transfer between the refrigerator and its surroundings follows a complex law \( Q \propto (\Delta T)^n \). Then

\[
Q_H = \int_0^\tau g_H(t) [T^n(t) - T_H^n] \, dt
\]

\[
Q_L = \int_0^\tau g_L(t) [T^n(t) - T_L^n] \, dt
\]

where \( g_H(t) \) and \( g_L(t) \) are heat conductivities between heat sink, heat source and working fluid, respectively. The heat conductivity is product of the overall heat transfer coefficient and corresponding heat transfer surface area of the heat exchanger. It is assume that at \( t = 0 \) the working fluid is in contact with high-temperature heat sink and is separated from the low-temperature heat source by an adiabatic boundary. At a later time \( t_1 (0 < t_1 < \tau) \), contact with the heat sink is broken and the working fluid is placed in contact with the heat source. Therefore, one has the flowing relationships

\[
g_H(t) = \begin{cases} g_H & 0 \leq t < t_1 \\ 0 & t_1 \leq t < \tau \end{cases}
\]

\[
g_L(t) = \begin{cases} 0 & 0 \leq t < t_1 \\ g_L & t_1 \leq t < \tau \end{cases}
\]

where \( g_H \) and \( g_L \) are constants.

From the first law of thermodynamics, the work input to the cycle is

\[
W = \int_0^\tau (g_H(t) [T^n(t) - T_H^n] - g_L(t) [T^n(t) - T_L^n]) \, dt
\]

From the second law of thermodynamics, the entropy change of the working fluid per cycle is

\[
\Delta S = \int_0^\tau \frac{1}{T(t)} [g_H(t) [T^n(t) - T_H^n] - g_L(t) [T^n(t) - T_L^n]] \, dt = 0
\]

Furthermore, since the heat capacity of the heat source is assumed to be constant, one has

\[
dQ_S = -CdT_s(t)
\]

Substituting Equation (2) into Equation (7) yields

\[
C \dot{T}_s(t) + g_L(t) [T^n(t) - T_L^n] = 0
\]

where \( \dot{T}_s(t) = dT_s(t)/dt \).

Our problem now is to determine the optimal configuration of the model cycle in which the minimum work input is needed under a given cycle duration \( \tau \) and cooling load \( Q_L \). Using Equations (5), (6) and (8), one has the modified Lagrangian

\[
L = g_H(t) [T^n(t) - T_H^n] - g_L(t) [T^n(t) - T_L^n] + \frac{\lambda}{T(t)} [g_H(t) [T^n(t) - T_H^n]]
\]

\[
- g_L(t) [T^n(t) - T_L^n] + \mu(t) \{C \dot{T}_s(t) + g_L(t) [T^n(t) - T_L^n]\}
\]
where \( \lambda \) is the Lagrangian constant, and \( \mu(t) \) is a function of time. The path for the working fluid which results in the minimum work for a given time interval \( \{0, \tau\} \) may now be obtained from the solution of the Euler–Lagrange equations. The Euler–Lagrange equations are given by

\[
\frac{\partial L}{\partial T(t)} - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{T}(t)} \right) = 0
\]

(10)

\[
\frac{\partial L}{\partial \dot{T}(t)} - \frac{d}{dt} \left[ \frac{\partial L}{\partial T(t)} \right] = 0
\]

(11)

For \( t_i \leq t < \tau \), substituting Equations (3), (4) and (9) into Equations (10) and (11), respectively, yields

\[
\lambda[T^*_n(t) - T^*(t)] + mnT^n(t)\{\lambda + [\mu(t) + 1]T(t)\} = 0
\]

(12)

\[
mng_z T^{n+1}_z(t)[T^*_n(t) - T^*(t)]^{n+1}[T(t) + \mu(t) - \lambda] = CT(t) \mu(t)
\]

(13)

From Equation (12) one can obtain

\[
\mu(t) = -\frac{\lambda}{T(t)} \frac{T^*_n(t) - T^*(t)}{mnT^{n+1}(t)} - 1
\]

(14)

The derivative of Equation (14) with respect to \( t \) is

\[
\dot{\mu}(t) = \frac{\lambda[-nT(t)T^*_n(t)\dot{T}_z(t) + (n + 1)\dot{T}(t)T^{n+1}_z(t) + (mn - 1)T^*_n(t)\dot{T}(t)T_z(t)]}{mnT^{n+1}(t)T_z(t)}
\]

(15)

Substituting Equations (8), (14) and (15) into Equation (13) yields

\[
n(m + 1)\dot{T}(t)[T^{n+1}_z(t)\dot{T}_z(t) - T^{n+1}_z(t)\dot{T}(t)] - (n + 1)\dot{T}(t)[T^*_n(t) - T^*(t)] = 0
\]

(16)

The solution of Equation (16) is

\[
[T^*_n(t) - T^*(t)]T(t)T^{n+1}(t) = a(mn)
\]

(17)

where \( a(mn) \) is a constant dependent on \( mn \).

Using the same way of calculation in the case of \( t_i \leq t < \tau \), one can obtain the relation of \( T(t) \) and \( T_H \) for \( 0 \leq t < t_i \)

\[
\lambda T^*_n(t) + \lambda(mn - 1)T^n(t) + mnT^{n+1}(t) = 0
\]

(18)

Equations (17) and (18) are the major results of this paper. They determine the relation between the temperatures of heat reservoirs and the working fluid. The heat source temperature \( T_z(t) \) and the working fluid temperature may be obtained by using Equations (8), (17) and (18), i.e. the optimal configuration of the refrigeration cycles.

4. Effects of heat transfer laws

(1). When \( n = 1 \), the heat transfer law becomes the generalized convective heat transfer law, Equations (17) and (18) become
\[ (T_s(t) - T(t))n = a(n), \quad t_i \leq t < \tau \]  
\[ \lambda T_{th} + \lambda(n - 1)T^n(t) + nT^{n+1}(t) = 0, \quad 0 \leq t \leq t_i \]  

where \( a(n) \) is a constant dependent on \( n \).

(i). If \( n = 1 \) further, Equations (25) and (26) are the results of the refrigeration cycle with Newton’s heat transfer law. Equations (17) and (18) become

\[ (T_s(t) - T(t))2 = a(1), \quad t_i \leq t < \tau \]  
\[ \lambda T_{th} + \lambda(1)T(t) + T^2(t) = 0, \quad 0 \leq t \leq t_i \]  

(ii). If \( n = 4 \), Equations (25) and (26) are the results of the refrigeration cycle with radiative heat transfer law. The temperatures of heat reservoirs and working fluid are complicate and follow the below relations

\[ (T_s^4(t) - T^4(t))n = a_2, \quad t_i \leq t < \tau \]  
\[ C T_s(t) = -g_s[T_s^4(t) - T^4(t)]^{5/4}, \quad 0 \leq t < t_i \]  

where \( a_2 \) is a constant.

(iii). If \( n = -1 \), Equations (25) and (26) are the results of the refrigeration cycle with linear phenomenological heat transfer law. Combining Equations (8), (25) and (26) gives
where \( a \) and \( b \) are two constants.

### 5. Fundamental optimal relation

Combining the change in the entropy of the working fluid heat absorbing process

\[
dS_L = C \ln(1 - Q_L / CT_L)
\]  

(30)

With the condition of internal reversibility, one can introduce an equivalent temperature of the heat source

\[
\dot{T}_L = -\frac{Q_L}{dS_L} = -\frac{Q_L}{C \ln(1 - Q_L / CT_L)}
\]  

(31)

And an equivalent temperature of working fluid in the heat absorbing process

\[
\dot{T}_{LC} = T_{HC} Q_L / Q_H
\]  

(32)

where \( T_{HC} \) is the temperature of working fluid at heat releasing process. Therefore, one can derive

\[
Q_L = g_L (T_L^* - T_L^w)^n (\tau - t_i)
\]  

(33)

\[
Q_H = g_H (T_{HC}^* - T_H^w)^n t_i
\]  

(34)

\[
W = Q_H - Q_L
\]  

(35)

\[
\varepsilon = Q_L / W = \dot{T}_{LC} / (T_{HC} - \dot{T}_{LC})
\]  

(36)

where \( \varepsilon \) is the COP of the refrigeration cycle.

Defining a ratio of period of two heat exchange processes (\( f \)) and the working fluid temperature ratio (\( x \)) as follows: \( f = t_i / (\tau - t_i) \), \( x = T_{LC}^* / T_{HC}^* \), where \( 0 \leq x \leq T_L^* / T_H^* \).

Combining Equations (31)-(36) gives the cooling load of the refrigeration cycle as

\[
R = Q_L / \tau = \frac{\tau f g_H}{(1 + f)(1 + \varepsilon)} \left( \frac{T_L^* - T_H^w [\varepsilon/(1 + \varepsilon)]^n}{[\varepsilon/(1 + \varepsilon)]^n + [f \varepsilon/(1 + \varepsilon)]^{(n+1)}} \right)^n
\]  

(37)

where \( r = g_H / g_L \). Taking the derivative of \( R \) with respective to \( f \) and setting it equal to zero yields

\[
f_o = r^{-\frac{n}{(n+1)}} [\varepsilon/(1 + \varepsilon)]^{(n-1)/(n+1)}
\]  

(38)

Substituting Equation (38) into Equation (37) yields
Equation (39) is another major result of this paper. It determines the optimal COP for the fixed cooling load. Since \( T^* \) in Equations (39) is a function of \( Q_L \), it is independent of \( Q_L \) only if \( C \) approaches infinite. If \( C \) approaches infinite, Equation (39) becomes the fundamental optimal relation between cooling load and COP of an endoreversible Carnot refrigerator coupled to infinite thermal capacity (constant-temperature) heat reservoirs with a complex law \( Q \propto (\Delta T^*)^n \) [34].

The relations between the optimal cooling load and COP with different values of \( m \) and \( n \) are shown in Figure 2. In the numerical calculations, \( T_H = 300K \), \( T_L = 260K \), \( C = 100J/(kg \cdot K) \) and \( g_H = g_L = 4W/K^m \) are set. One can see that the heat transfer law has significant influences on the optimal relation between cooling load and COP of the generalized endoreversible refrigerator cycle. The cooling load is a monotonic decreasing function of COP when \( n > 0 \) and \( m > 0 \), and a parabolic-like curve when \( n < 0 \) and \( m > 0 \).

![Figure 2. The optimal relation between cooling load and COP of refrigerator with different heat transfer laws](image)

6. Conclusion
The optimal configuration and performance of a refrigeration cycle operating between a finite low-temperature source and an infinite high-temperature sink is studied. In the refrigeration model the only irreversibility of finite rate heat transfer is considered. The heat transfer obeys a complex heat transfer law, including Newtonian heat transfer law, linear phenomenological heat transfer law, radiative heat transfer law, DuLong-Petit heat transfer law, generalized convective heat transfer law and generalized radiative heat transfer law, \( Q \propto \Delta(T^*)^n \). The heat transfer law has the significant influence on the optimal relation between cooling load and COP of the generalized endoreversible refrigerator. The cooling load is a monotonic decreasing function of COP when \( n > 0 \) and \( m > 0 \), and a parabolic-like curve when \( n < 0 \) and \( m > 0 \). The obtained results include those with various heat transfer laws and infinite low-temperature source, and can provide some theoretical guidelines for the designs of practical refrigerators.

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References


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