Primary instability regions of square plates subjected to uniaxial in-plane periodic compression load

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Abstract
The purpose of this paper is to investigate the dynamical behavior and the parametric primary instability of thin square elastic plate subjected to uniform in-plane uniaxial periodic compression load. The plate is assumed to be simply supported along its four edges (SSSS). Based on small deflection theory of plates, the value of natural frequency of free vibration and buckling load are determined by assuming the mode shape of vibration of plates simply supported along all edges. If the plate subjected to parametric excitation the plate lose its stability dynamically by flutter. The instability regions are found according to Bolotin's concept. In the present research the effects of excitation parameter and load value are studied. It is found that increasing both of the excitation parameter and loading value cause an increase in the regions of instability. An experimental work is conducted to verify the accuracy of the mathematical model. Good agreement between the theoretical and experimental results is obtained within 8% error.

Keywords: Parametric instability; Small deflection theory; Buckling load, Primary instability regions; Excitation parameter.

1. Introduction
Thin plates as shown in Figure 1 are widely used in all fields of engineering which are subjected to static or dynamic loads like, mechanical, civil, aerospace engineering, architectural structures, marine, solar panels, bridges, hydraulic structures, containers, airplanes, missiles, ships, instruments and machine parts.
Plates used in structure elements which subjected to in-plane periodic compression loads (dynamic load) may undergo unstable transverse vibrations, leading to a parametric resonance, due to certain combinations of the values of load parameters and natural frequency of transverse vibration. The study of the static and dynamic behavior of plate under the in-plane compression load is of certain importance, because this load affects the response of the plate. If the plate subjected to static in-plane compression load, it may be unstable statically. As the magnitude of this load increases, the natural frequency of vibration reduces and at the critical buckling load the natural frequency failed (becomes zero). If the plate subjected to a periodic in-plane compression dynamic loads, it will losses its stability dynamically in the form of resonant transverse vibrations. This phenomenon is known as "parametric resonance".
The study of dynamic instability started at (1831). This phenomenon was first observed by Faraday, when he noticed that when a fluid filled container vibrates vertically, fluid surface oscillates at half the frequency of the container.

K. Takahashu, et al., [1], determined the natural frequency of free vibration, static buckling load and boundaries of regions of dynamic instability of a cantilever rectangular plate. Thin plate of small deflection theory was used to derive the governing equation of plates. The plate assumed to be subjected to an in-plane sinusoidally varying load applied along the free end. The vibration and buckling problems were solved according to Rayleigh-Ritz method, while the dynamic stability problem according to Hamilton's principle to derive time dependent variables. Unstable regions of dynamic instability are given for various loading conditions. Simple parametric resonances and combination parametric resonances are acquired for various loading conditions. Numerical results are offered for several example problems, and they show that the above method is reasonably accurate.

A. K. L. Srivastava, et al., [3], used the finite element method to study vibration and dynamic instability behavior of stiffened plate under the effect of in-plane periodic edges load. Various boundary conditions, varying mass and stiffness properties, aspect ratios and varying number of stiffeners have been discussed for dynamic instability using the method of Hill's infinite determinants. The results explain that the principal instability regions have significant effect.

A. K.L. Srivastava, et al., [6], used the finite element method to study vibration and buckling characteristics of stiffened plates under the effect of in-plane uniform and non-uniform edge loading. Various boundary conditions, aspect ratios, varying mass and stiffness properties and varying number of stiffeners have been analyzed for buckling and vibration studies. The equation of motion used to determine the characteristic equations for the natural frequencies, buckling loads and their corresponding mode shapes. This study also include the effects of aspect ratio, boundary conditions, the position of stiffeners and number of stiffeners parameters upon the buckling load parameter and fundamental natural frequency of the stiffened plates. The above results solutions give good agreement with other solution.

A. K. L. Srivastava, et al., [3], used the finite element method to study vibration and dynamic instability of shear deformable plates. Also experimental work is added to verify the accuracy of the theoretical work.

In the present work, the primary instability regions of square plates are investigated using exact solution. Also experimental work is added to verify the accuracy of the theoretical work.

2. Theoretical analysis
2.1 Based equation
When a load applied on a plate is parametric (static plus dynamic components), the plate will resonate (fluttering motion) in regions known as regions of instability. The parametric excitation load can be defined by [8]:

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By neglecting the effect of longitudinal vibration in the middle plane, then only the transverse vibration is of important interest and taking into account. Also neglecting the effects of shear deformation and rotary inertia, then the basic equation of vibration subjected to in-plane parametric load can be summarized as [8]:

\[
D \frac{d^4w}{dt^4} + \rho h \frac{\partial^2 w}{\partial t^2} = -(N_s + \delta N_s \cos \theta t) \frac{\partial^2 w}{\partial x^2}
\]  

(2)

2.2 Primary regions of instability
The primary regions of instability can be evaluated when the period of time is taken (2T) and the summation takes the odd values. If the plate is simply supported along all edges then the solution of Eq. (2) is written in the form [8]:

\[
W_{km}(x, y, t) = A_{km} \sin \frac{knx}{a} \sin \frac{mny}{b} f_{km}(t)
\]

(3)

where:

\( A_{km} \): Constants.

\( f_{km}(t) \): Unknown function of time.

\( k, m \): Number of half waves in the axial and transverse directions respectively.

Substituting Eq.(3) into Eq.(2) gives:

\[
\rho h \frac{d^2f}{dt^2} + D \left( \frac{k^2 \pi^2}{a^2} + \frac{m^2 \pi^2}{b^2} \right) f - (N_s + \delta N_s \cos \theta t) \frac{k^2 \pi^2}{a^2} f = 0
\]

(4)

After some mathematical manipulations, Eq.(4) can be written as:

\[
\frac{d^2f}{dt^2} + \omega_{km}^2 \left[ 1 - \left( \frac{N_s + \delta N_s \cos \theta t}{N_{cr}} \right) \right] f = 0
\]

(5)

where:

\( N_{cr} \): Euler's buckling load (N).

\( \omega_{km} \): Natural frequency of free vibration (rad/sec).

Equation.(5) is written in the form:

\[
\frac{d^2f}{dt^2} + \omega_s^2(1 - 2 \varphi_{km} \cos \theta) = 0
\]  

(6)
where: $\omega$: Frequency of vibration of plates subjected to uniaxial in-plane compression static force (rad/sec).

$$\omega_s = \omega_{km} \sqrt{1 - \frac{N_s}{N_{cr}}}$$  \hspace{1cm} (7a)

$$\varphi_{km} = \frac{\delta N_s}{2(N_{cr} - N_s)}$$  \hspace{1cm} (7b)

The solution of Eq.(6) can be used to find the regions of dynamic instability, the function represent these regions is expressed by [8];

$$f(t) = \sum_{m=1,3,5,...}^{\infty} \left( \psi_m \sin \frac{m\theta t}{2} + \phi_m \cos \frac{m\theta t}{2} \right)$$  \hspace{1cm} (8)

where: $\psi_m, \phi_m$: Unknown coefficient.

Substitute Eq.(8) into Eq.(6) results in the following equation:

$$-2 \varphi_{km} \omega_s^2 \psi_m \sin \frac{m\theta t}{2} \cos \theta t - 2 \varphi_{km} \omega_s^2 \phi_m \cos \frac{m\theta t}{2} \cos \theta t = 0$$  \hspace{1cm} (9)

Using the trigonometric relations and equating the coefficients of $(\sin (m\theta t/2))$ and $(\cos (m\theta t/2))$ Eq.(9) leads to the following two equations in matrix form:

$$\begin{bmatrix} 1 + \varphi_{km} - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & 0 & \vdots & \psi_1 \\ -\varphi_{km} & 1 - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & \vdots & \psi_2 \\ 0 & -\varphi_{km} & 1 - \frac{25\theta^2}{4\omega_s^2} & \vdots & \psi_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{bmatrix} = 0$$  \hspace{1cm} (10)

$$\begin{bmatrix} 1 - \varphi_{km} - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & 0 & \vdots & \phi_1 \\ -\varphi_{km} & 1 - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & \vdots & \phi_2 \\ 0 & -\varphi_{km} & 1 - \frac{25\theta^2}{4\omega_s^2} & \vdots & \phi_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \vdots \end{bmatrix} = 0$$  \hspace{1cm} (11)

The order of the above two matrix equations is infinite and the size of the matrices is the same. Because of the solution related to the period $2T$ exists, then applying the condition of the existence of nontrivial solution (the determinant of the two matrices must be equated to zero). These two matrices may be combined together under the $\pm$ sign to calculate the determinant, in which the following equation results:

$$\begin{bmatrix} 1 \pm \varphi_{km} - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & 0 & \vdots & \psi_1 \\ -\varphi_{km} & 1 - \frac{\theta^2}{4\omega_s^2} & -\varphi_{km} & \vdots & \psi_2 \\ 0 & -\varphi_{km} & 1 - \frac{25\theta^2}{4\omega_s^2} & \vdots & \psi_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \vdots \end{bmatrix} = 0$$  \hspace{1cm} (12)

According to Bolotin's concept [8], using the first approximation, only the first diagonal element of the above matrix is taken and solved to give the boundaries of the upper and lower limits of the primary regions of dynamic instability;
1 ± \( \phi_{km} \) - \( \frac{\theta^2}{4\omega_x^2} \) = 0 \hspace{1cm} (13)

Substitute Eqs. (7a) and (7b) into Eq.(13) gives:

\[ \frac{\theta^2}{4\omega_{km}^2} = 1 - \phi + \frac{\delta}{2} \phi \] \hspace{1cm} (14)

where: \( \phi = \frac{N_s}{N_0} \)

Equation (14) can be separated into two equations:

\[ \frac{\theta^2}{4\omega_{km}^2} = 1 - \phi + \frac{\delta}{2} \phi \] \hspace{1cm} (15)

\[ \frac{\theta^2}{4\omega_{km}^2} = 1 - \phi - \frac{\delta}{2} \phi \] \hspace{1cm} (16)

Equation (15) used to determine the upper limit of the primary instability region while Eq.(16) used to determine the lower limit.

3. Experimental work

3.1 Mechanical properties of plates

The tensile test specimens are picked from the tested plate (stainless steel 304) according to ASTM Number (A370-2012). This test is used to determine the tensile strength, yield strength and elongation of plates. This test was done in the standardization and quality control center gives the properties shown in Table 1.

<table>
<thead>
<tr>
<th>Properties</th>
<th>Values</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tensile strength</td>
<td>494</td>
<td>N/mm²</td>
</tr>
<tr>
<td>Yield strength</td>
<td>300</td>
<td>N/mm²</td>
</tr>
<tr>
<td>Elongation% (L₀ =80mm)</td>
<td>26</td>
<td>-</td>
</tr>
<tr>
<td>Modulus of elasticity</td>
<td>183</td>
<td>GPa</td>
</tr>
</tbody>
</table>

3.2 In-plane parametric compression load

To apply the in-plane periodic compression load, the shaft which prepared for this purpose is constructed so that different amplitudes of excitation can be satisfied. The amplitude of excitation is controlled by shifting the shaft through the slot as illustrated in Figure 2. When the shaft imposed at slot center, the amplitude of excitation is zero (no excitation), while the maximum excitation is improved when the shaft imposed at the end of the slot.

Figure 2. Mechanism of creating the amplitude of excitation.
3.3 Accelerometer measurements
A 3-axis accelerometer type ADXL335 and a microcontroller was used to receive the response of tested model (amplitude of vibration). The accelerometer measures the transverse acceleration of the plate due to dynamic load in the form of voltage signal. This accelerometer is connected to the computer through a microcontroller interface.

The signal that picked from the accelerometer of the tested model which is processed by the microcontroller is a voltage signal. A calibration is carried out to convert the readings to amplitude.

3.3.1 Calibration of accelerometer
Calibration is done according to the relations shown below [9] and displayed on computer in the form of acceleration reading.

\[
v_{ref} = \frac{v_1 + v_2}{2} \quad (17)
\]

\[
se = \frac{v_1 - v_2}{2g} \quad (18)
\]

\[
acc = \frac{v_{in} - v_{ref}}{se} - g \quad (19)
\]

3.4 Stability test
Table 2 shows the properties of the plates using in the stability test and Figure 3 shows the rig used in stability test. Figure 4 shows the flow chart of the testing rig. This study confirms the effect of periodical load and static load (spring under prescribed compression) on dynamic stability. To measure the frequency at which the plate loses its stability by flutter, the accelerometer is placed at the center of edge under excitation. The frequency of excitation increased until maximum vibration flutter is achieved. Figure 3 show the flowchart of stability test.

This test is repeated for three values of excitation parameter (\(\delta = 0.129, 0.288, 0.433\)) so this gives three values of fluttering frequencies for the upper limit and three values of fluttering frequencies for the lower limit. These values of fluttering frequencies form the regions of primary instability.

4. Results and discussions
4.1 Theoretical results
The values of limits of the primary instability regions depend on the excitation parameter (\(\delta\)), load ratio (ratio of static load to bucking load (\(\phi\))) and natural frequency of free vibration. The primary instability regions shifted to higher when the excitation increase, also the instability regions decreased with decreased the excitation parameter. The instability regions also increase with increase the natural frequency and decrease with decrease the natural frequency as you seen in eqs.(15) and (16). This attributed due to effect of stiffness on the mode of vibration. The enclosed area between the upper and lower limits refers to unstable regions as shown in Figure 5.

To investigate the effect of the excitation parameter (\(\delta\)) on regions of instability, this parameter is changed in a suitable range while the other affecting parameters are fixed at suitable values. In general the excitation parameter \(\delta\) is changed from 0 to 0.5 and the value of nondimensional load is fixed at 0.3 (\(\phi = 0.3\)). It is shown that the width of instability regions increased with increasing \(\delta\) as shown in Figure 6. This is attributed to the fact that the energy transferred from the load source to the plate increased with increasing the excitation parameter. This transferred energy will be increased as the frequency is increased because the amplitude of vibration is constant. Also one can deduce from this figure that the region of the instability increases for higher modes due to effect of stiffness on the mode of vibration.

Table 2. Properties of the plates used in the experimental work.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length (a)</td>
<td>0.3</td>
<td>M</td>
</tr>
<tr>
<td>Width (b)</td>
<td>0.3</td>
<td>M</td>
</tr>
<tr>
<td>Thickness (h)</td>
<td>0.0008</td>
<td>M</td>
</tr>
<tr>
<td>Density ((\rho))</td>
<td>7800</td>
<td>Kg/m³</td>
</tr>
<tr>
<td>Modulus of Elasticity (E)</td>
<td>183</td>
<td>GPa</td>
</tr>
</tbody>
</table>
The effect of the in-plane compression fluctuating load on primary instability regions is illustrated in Figure 7. In this figure the value of loading is changed in the range 0-0.5 and the value of excitation parameter is fixed at 0.3. As shown in this figure, the band of frequencies (between upper and lower limits) that the plate loses its stability by flutter increased as the load increased. This behavior can be attributed to the time period of the harmonic fluctuation of the uniaxial in-plane compression load occurs at period 2T. The flutter that appears at this period can be caused by increasing the load causing an increase in the inertia forces delivered in the structure; therefore the size of the unstable region becomes wider. Also one can deduce from this figure that the region of the instability increases for higher modes due to effect of stiffness on the mode of vibration.

Figure 3. Stability testing rig.

Figure 4. Flow chart of the testing rig.
Figure 5. Primary instability region of square plate.

Figure 6. Effect of excitation parameter on primary instability regions of SSSS plate, $\varnothing = 0.3$.

Figure 7. Effect of excitation parameter on primary instability regions of SSSS plate, $\delta = 0.3$. 
4.2 Experimental results
The experimental verification on the dynamic stability of square plate simply supported along all edges is carried out to check the accuracy of the theoretical values. The properties of plates used in this study are listed in Table 2. The stiffness of spring used to apply static load is \( k_s = 2368.134 \text{ N/m} \) and the stiffness of the spring that used to apply dynamic load is \( k_d = 1367.514 \text{ N/m} \). The experimental results may be summarized as follows:

1. The effect of excitation parameter \( (\delta = 0.129, 0.288, 0.433) \) on the regions of dynamic instability of square plate is illustrated in Figure 8. This study is done by fixing the value of the load \( (N_s = 47.362 \text{ N}) \) where a spring used to applied static load is compressed at distance 2cm). The differences between experimental results and theoretical results gives good agreement with error less than 7%, Table 3. Figure 9 shwos the acceleration curve (amplitude of vibration) of experimental primary instability regions.

2. The effect of loading \( (N_s = 47.362, 94.075, 139.135) \) on the theoretical and experimental regions of dynamic instability of square plate are illustrated in Figure 10. This experiment is done by fixing the value of the excitation parameter at \( (\delta = 0.129) \). The differences between experimental results and theoretical results gives good agreement with error less than 8% as shown in Table 4. The differences between anlytical results and experimental works due to many causes conclude, the mass of accelerometer is neglected, neglect the effect of shear strain in the xz plane and zy plane, neglected the effect of normal stress in z- direction and neglected the effect of structure damping.

![Figure 8. Theoretical and experimental effects of excitation parameter on primary instability region of SSSS plate.](image)

<table>
<thead>
<tr>
<th>Theoretical flutter frequency (rad/sec)</th>
<th>Experimental flutter frequency (rad/sec)</th>
<th>Differences %</th>
</tr>
</thead>
<tbody>
<tr>
<td>510</td>
<td>476</td>
<td>6.666</td>
</tr>
</tbody>
</table>

Table 3. Comparison between experimental and theoretical results of flutter frequency due to the effect of excitation parameter \( N_s = 47.362 \text{ N} \).
Figure 9. Acceleration curve for dynamic instability with excitation parameter ($\delta = 0.433$).

![Acceleration curve](image)

Figure 10. Theoretical and experimental effects of loading on primary instability region of SSSS plate.

![Theoretical and experimental effects](image)

Table 4. Comparison between experimental and theoretical results of flutter frequency due to the effect of loading $\delta = 0.129$.

<table>
<thead>
<tr>
<th>Theoretical flutter frequency (rad/sec)</th>
<th>Experimental flutter frequency (rad/sec)</th>
<th>Differences %</th>
</tr>
</thead>
<tbody>
<tr>
<td>504</td>
<td>468</td>
<td>7.14</td>
</tr>
</tbody>
</table>

5. Conclusions
Throughout the present study the following conclusion can be withdrawn:
1. The band of frequencies that the plate loses its stability by flutter increased with increasing of the excitation parameter.
2. The band of frequencies that the plate loses its stability by flutter increased with increasing of the load value.
3. The regions of instability increase as the mode of vibration increases.

Acknowledgements
The authors would like to thank the Departments of Mechanical Engineering, Karbala University for supporting tests facilities of this study.
Nomenclatures

<table>
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<th>Definitions</th>
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<tr>
<td>m/s</td>
<td>Acceleration</td>
<td>N</td>
</tr>
<tr>
<td>mm</td>
<td>Length of plate</td>
<td>N/m</td>
</tr>
<tr>
<td>N/m</td>
<td>Flexure rigidity</td>
<td>m/s^2</td>
</tr>
<tr>
<td>mm</td>
<td>Thickness of plate</td>
<td>N</td>
</tr>
<tr>
<td>s^2</td>
<td>Sensitivity</td>
<td>V</td>
</tr>
<tr>
<td>V</td>
<td>Output voltage of accelerometer</td>
<td>V</td>
</tr>
<tr>
<td>V</td>
<td>Reference voltage</td>
<td>V</td>
</tr>
<tr>
<td>V</td>
<td>Upper voltage</td>
<td>V</td>
</tr>
<tr>
<td>V</td>
<td>Lower voltage</td>
<td>V</td>
</tr>
<tr>
<td>Mm</td>
<td>Displacement in z direction</td>
<td>V</td>
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Greek symbols

<table>
<thead>
<tr>
<th>Symbols</th>
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<tbody>
<tr>
<td>δ</td>
<td>Excitation parameter</td>
<td></td>
</tr>
<tr>
<td>θ</td>
<td>Excitation frequency</td>
<td>rad/sec</td>
</tr>
<tr>
<td>ρ</td>
<td>Density of plate</td>
<td>kg/m^3</td>
</tr>
<tr>
<td>ωs</td>
<td>Frequency of vibration of plates</td>
<td>rad/sec</td>
</tr>
<tr>
<td>ωs</td>
<td>Subjected to uniaxial in-plane</td>
<td></td>
</tr>
<tr>
<td></td>
<td>compression static force</td>
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</table>

References


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